# Integration of Complete System of Dynamic Equations for Ideal Fluid

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**Abstract**

The Eulerian system of dynamic equations for the ideal (nondissipative) fluid is closed but incomplete. The complete system of dynamic equations arises after appending Lin constraints which describe motion of fluid particles in a given velocity field. The complete system of dynamic equations fo the ideal fluid can be integrated. Description in terms of hydrodynamic potentials (DTHP) arises as a result of this integration. The integrated system contains indefinite functions of three arguments, which can be expressed via initial and boundary conditions. The remaining initial and boundary conditions for the integrated system can be made universal (i.e. the same for all fluid flows), and the resulting system of equations contains full information about the fluid flow including initial and boundary conditions for the fluid flow. Some hydrodynamic potentials appear to be frozen into the fluid, and the Kelvin's theorem on the velocity circulation can be formulated in a contour-free form. Description in terms of the wave function (DTWF) appears to be a kind of DTHP. Calculation of slightly rotational flows can be carried out on the basis of DTHP, or DTWF. Such a description of a rotational flow appears to be effective.

## 1 Introduction

Dynamic equations for an ideal (nondissipative) fluid are written conventionally in the form:

$$\frac{\partial\rho}{\partial t}+\nabla(\rho\mathbf{v})=0 \tag{1.1}$$

$$\frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}\nabla)\mathbf{v}=-\frac{1}{\rho}\nabla p,\qquad p=\rho^2\frac{\partial E}{\partial\rho} \tag{1.2}$$

$$\frac{\partial S}{\partial t}+(\mathbf{v}\nabla)S=0 \tag{1.3}$$

where dependent variables $\rho$ and $\mathbf{v}=\{v^1,v^2,v^3\}$, $S$ are respectively the fluid mass density, the fluid velocity and entropy per unit mass considered as functions of independent Eulerian variables $x = \{t, \mathbf{x}\}$. $p$ is athe pressure, and $E = E(\rho, S)$ is an internal energy of an unite mass considered as a function of $\rho$ and $S$. The internal energy $E = E(\rho, S)$ is an unique characteristic of the ideal fluid.

The system of hydrodynamic equations (1.1)–(1.3) is a closed system of differential equations which has an unique solution inside some space-time region $\Omega$, provided dependent dynamic variables $\rho$ and $\mathbf{v} = \{v^1, v^2, v^3\}$, $S$ are given as functions of three arguments on the space-time boundary $\Gamma$ of the region $\Omega$. Being closed, the system (1.1)–(1.3) describes nevertheless only momentum-energetic characteristics of the fluid. Motion of the fluid particles along trajectories is described by so called Lin (1963) constraints

$$\frac{\partial \boldsymbol{\xi}}{\partial t} + (\mathbf{v}\nabla)\boldsymbol{\xi} = 0, \tag{1.4}$$

where quantities $\boldsymbol{\xi} = \boldsymbol{\xi}(t, \mathbf{x}) = \{\xi_\alpha(t, \mathbf{x})\}$, $\alpha = 1, 2, 3$ label fluid particles. They will be referred to as particle labeling (curvilinear Lagrangian coordinates). If the equations (1.4) are solved and $\boldsymbol{\xi}$ is determined as a function of $(t, \mathbf{x})$, the finite relations

$$\boldsymbol{\xi}(t, \mathbf{x}) = \boldsymbol{\xi}_{\text{in}} = \text{const} \tag{1.5}$$

describe implicitly a fluid particle trajectory and a motion along it.

The system of eight equations (1.1) –(1.4) forms a complete system of dynamic equations describing a fluid, whereas the system of five equations (1.1)–(1.3) forms a curtailed system of dynamic equations. The last system is closed, but to be a complete system, it must be supplemented by the kinematic equations

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}(t, \mathbf{x}), \qquad \mathbf{x} = \mathbf{x}(t, \boldsymbol{\xi}) \tag{1.6}$$

where $\mathbf{v}(t, \mathbf{x})$ is a solution of the system (1.1)-(1.3). Three equations (1.6) are equivalent to (1.4), because any solution $\boldsymbol{\xi} = \boldsymbol{\xi}(t, \mathbf{x})$ of (1.4) is a set of three integrals of equations (1.6).

There is a lack of understanding of the fact that the Euler system (1.1)–(1.3) is a curtailed one and that the equations (1.4) describe a motion of fluid particles. Even Lin (1963) who introduced equations (1.4) considered them "as the condition for the conservation of the identity of particles", but not as kinematic equations describing a motion of fluid particles in a given velocity field. Betherton (1970) investigated the Lin constraints and showed a connection between them and the Kelvin's theorem on the velocity circulation. But the very important statement that the Lin constraints (1.4) are a kind of kinematic equations (1.6) was not mentioned directly. In general, we failed to find in literature any reference to equations (1.4) as necessary kinematic equations, although this fact seems to be evident. Apparently, it is connected with the common belief that the Lin constraints (1.4) are useless, if one is interested only in the velocity field of the flow.

We should like to show a necessity of these equations in a very simple example, modelling a situation in the hydrodynamics in a grotesque form. Let us consider a particle moving in uniform gravitational field $\mathbf{g}$ =const. Dynamic equations have the form

$$\dot{\mathbf{v}} = \mathbf{g}, \qquad \dot{\mathbf{x}} = \mathbf{v}, \qquad \mathbf{g} = \text{const} \tag{1.7}$$

where $\mathbf{x}$ and $\mathbf{v}$ are functions of $t$ describing respectively position and velocity of the particle. First three equations (1.7) constitute a closed subsystem [analog of the Euler system (1.1)–(1.3)] of the full system of six equations. Equations of the subsystem can be solved independently of the remaining equations. But it does not mean that this closed subsystem may be considered as a system of dynamic equations describing a particle, even if we are interested only in momentum-energetic properties of the particle. Such important characteristic of the particle as the energy integral $\mathbf{v}^2/2-\mathbf{gx} = \text{const}$ cannot be derived on the base of only first three equations (1.7). This integral is a sum of integrals

$$(v_\alpha)^2/2 - g_\alpha x_\alpha = C_\alpha = \text{const}, \qquad v_\alpha = \frac{dx^\alpha}{dt}, \qquad \alpha = 1,2,3 \tag{1.8}$$

These integrals are analogs of integrals of the complete system (1.1)-(1.4) [see below (1.11)]. This example shows that a closed system of equations and a complete system of dynamic equations is not the same. The Lagrangian formulation of hydrodynamic equation includes equations (1.4) in the form (1.6) automatically. Lagrangian formulation is equivalent to the Eulerian formulation (1.1)–(1.3), provided equations (1.4) are appended to it.

Equations (1.1), (1.3) can be integrated on the basis of (1.4) in the form

$$S(t,\mathbf{x}) = S_0(\boldsymbol{\xi}) \tag{1.9}$$

$$\rho(t,\mathbf{x}) = \rho_0(\boldsymbol{\xi}) \frac{\partial(\xi_1,\xi_2,\xi_3)}{\partial(x^1,x^2,x^3)} \equiv \rho_0(\boldsymbol{\xi}) \frac{\partial(\boldsymbol{\xi})}{\partial(\mathbf{x})} \tag{1.10}$$

where $S_0(\boldsymbol{\xi})$ and $\rho_0(\boldsymbol{\xi})$ are arbitrary integration functions of the argument $\boldsymbol{\xi}$. These functions can be determined from the initial (and boundary) conditions. Three equations (1.2) also can be integrated on the basis of (1.4). These integrals have the form

$$\mathbf{v}(t,\mathbf{x}) = \mathbf{u}(\varphi,\boldsymbol{\xi},\eta,S) \equiv \nabla\varphi + g^\alpha(\boldsymbol{\xi})\nabla\xi_\alpha - \eta\nabla S, \tag{1.11}$$

where $g(\boldsymbol{\xi}) = \{g^\alpha(\boldsymbol{\xi})\}$, $\alpha = 1,2,3$ are arbitrary integration functions of argument $\boldsymbol{\xi}$, and $\varphi$, $\eta$ are new dependent variables satisfying dynamic equations

$$\frac{\partial\varphi}{\partial t} + \mathbf{u}(\varphi,\boldsymbol{\xi},\eta,S)\nabla\varphi - \frac{1}{2}[\mathbf{u}(\varphi,\boldsymbol{\xi},\eta,S)]^2 + \frac{\partial(\rho E)}{\partial\rho} = 0 \tag{1.12}$$

$$\frac{\partial\eta}{\partial t} + \mathbf{u}(\varphi,\boldsymbol{\xi},\eta,S)\nabla\eta = -\frac{\partial E}{\partial S}. \tag{1.13}$$

If five dependent variables $\varphi$, $\boldsymbol{\xi}$, $\eta$ satisfy the system of equations (1.4), (1.12), (1.13), the five dynamic variables $S$, $\rho$, $\mathbf{v}$ (1.9)–(1.11) satisfy dynamic equations (1.1)–(1.3). Indefinite functions $g^\alpha(\boldsymbol{\xi})$ can be determined from initial and boundary conditions in such a way that the initial and boundary conditions for variables $\varphi$, $\boldsymbol{\xi}$, $\eta$ were universal in the sense that they do not depend on the fluid flow.

According to (1.10), (1.11) the physical quantities $\rho$, $\mathbf{v}$ are obtained as a result of differentiation of the variables $\varphi$, $\boldsymbol{\xi}$, $S$, and the variables $\varphi$, $\boldsymbol{\xi}$, $\eta$ can be regarded as hydrodynamic potentials. These potentials appear in the Hamilton fluid dynamics

(Salmon, 1988) as dependent variables. The hydrodynamic potentials arise by a natural way. They associate with the name of Clebsch (1857, 1859) who introduced these quantities for the incompressible fluid. Such quantities as $g^{\alpha}(\boldsymbol{\xi})$ also appear in the Hamilton fluid mechanics (Salmon,1988), but they appear as dependent variables (Lagrage invariants) satisfying dynamic equations of the type (1.4). They also are regarded as hydrodynamic potentials. Note that in the Hamilton fluid mechanics the quantities $g^{\alpha}$ are considered simply as dependent variables, but not as indefinite functions of $\boldsymbol{\xi}$, arising as a result of integration, although corresponding dynamic equations for $g^{\alpha}$ can be integrated easily.

It should distinguish between the integration and a change of variables which does not contain arbitrary functions explicitly. For instance, let us substitute $g^{\alpha}(\boldsymbol{\xi})$, $\alpha = 1, 2, 3$ by new dependent variables $A^{\alpha}$, $\alpha = 1, 2, 3$, imposing on them constraints

$$\frac{\partial A^{\alpha}}{\partial t} + \mathbf{u}(\varphi, \boldsymbol{\xi}, \eta, S)\nabla A^{\alpha} = 0, \qquad \alpha = 1, 2, 3. \tag{1.14}$$

Then one has instead of (1.11)

$$\mathbf{v}(t, \mathbf{x}) = \mathbf{u}(\varphi, \boldsymbol{\xi}, \eta, S) \equiv \nabla\varphi + A^{\alpha}\nabla\xi_{\alpha} - \eta\nabla S, \tag{1.15}$$

The eighth order system (1.4), (1.12)–(1.15) arises instead of the fifth order system (1.4), (1.11)–(1.13). This system is not an integrated one, because it has the higher order and contains only two arbitrary functions (1.9), (1.10). Different modifications of such kind systems were derived (Salmon, 1988). They cannot be considered as integrated systems. Of course, one can easily integrate equtions (1.14) on the basis of (1.4), and return to (1.11). But until this is not made and the number of dynamic variables is not decreased, the system cannot be considered as integrated.

The integration of the complete system (1.1)-(1.4) and some corollaries of this integration correlates with the Hamilton properties of the ideal fluid [Harivel, (1955); Eckart, (1960); Seliger and Whithem, (1967), Salmon, (1988); Zakharov and Kuznetsov (1997)]. It is connected with the fact that the curtailed system (1.1)-(1.3) is not a Hamiltonian system in itself, whereas the complete system (1.1)-(1.4) is a Hamiltonian one that can be easily seen in the example (1.7). Constructing Hamiltonian mechanics of the ideal fluid, one uses (implicitly or explicitly) the Lin constraints (or part of them). It is this expansion of the curtailed system (but not Hamiltonian properties) that is important for integration and derivation of other useful results. To show this, the Hamiltonian technique and Hamiltonian properties of the ideal fluid will not be used at all.

Note that the curtailed system (1.1)–(1.3) has the same order as the integrated system (1.4), (1.12), (1.13), but it takes into account neither initial conditions, nor kinematic equations (1.6). The fact that the ideal fluid considered as a dynamic system admits both the curtailed system (1.1)–(1.3) and the integrated system (1.4), (1.12), (1.13) is connected closely with the group of the relabeling transformation (relabeling group)

$$\xi_{\alpha} \rightarrow \tilde{\xi}_{\alpha} = \tilde{\xi}_{\alpha}(\boldsymbol{\xi}), \qquad D \equiv \det \parallel \partial\tilde{\xi}_{\alpha}/\partial\xi_{\beta} \parallel \neq 0, \qquad \alpha, \beta = 1, 2, 3 \tag{1.16}$$

$$\varphi = \xi_0 \to \tilde{\xi}_0 = \tilde{\varphi} = \tilde{\xi}_0(\xi_0) + a_0(\boldsymbol{\xi}), \qquad \partial\tilde{\xi}_0/\partial\xi_0 > 0 \tag{1.17}$$

where $\xi = \{\xi_0, \boldsymbol{\xi}\}$ are curvilinear Lagrangian coordinates in the space-time, $\tilde{\xi} = \{\tilde{\xi}_0, \tilde{\boldsymbol{\xi}}\}$ is another system of curvilinear Lagrangian coordinates. $\tilde{\boldsymbol{\xi}}_0$ and $a_0$ are arbitrary functions of $\boldsymbol{\xi}$. $\tilde{\xi}_0$ is arbitrary function of $\xi_0$. $\xi_0$ is a temporal coordinate, and $\boldsymbol{\xi}$ are spatial ones.

The relabeling group is a symmetry group of the fluid considered as a dynamic system. This circumstance admits to integrate the complete system (1.1)–(1.4). Any special particle labeling is unessential from physical viewpoint. It is a reason why several equations (1.1)–(1.3) of the complete system form a closed system describing conservation laws. The relabeling group is used in hydrodynamics comparatively recently Eckart (1938, 1960), Calkin (1963), Bretherton (1970), Friedman and Schutz (1978), Salmon (1982), Zacharov and Kuznetsov (1997) and others.

The integrated system (1.4), (1.12), (1.13) looks more complicated, than the curtailed system (1.1)–(1.3). It is quite natural, because the integrated system is a complete system which contains five indefinite functions $\rho_0$, $S_0$, $g^\alpha$ describing initial and boundary conditions.

There is a common belief that the curtailed Eulerian system (1.1)–(1.3) is sufficient for calculating fluid flows, and a use of the complete system (1.1)–(1.4) is not necessary in most of cases. In general, there is a puzzling question of such a kind. If both the Euler system (1.1)-(1.3) and the integrated system (1.4), (1.12), (1.13) have the same order and the same number of dependent variables, why should one consider the integrated system (1.4), (1.12), (1.3) which looks more complicated, than the Euler system? The answer is as follows. Although the Lin constraints describe mainly a motion of fluid particles in the given velocity field, nevertheless the integrated system contains additional information which is necessary for a calculation of rotational fluid flows. This additional information concerns fluid properties described by the Kelvin's theorem on the velocity circulation. This theorem is an *attribute of the complete system* (1.1)-(1.4), because it refers to the contour connected rigidly with the fluid particles moving according to dynamic equations (1.6), or (1.4). In general, the Kelvin's theorem cannot be formulated only in terms of the velocity field. On the other hand, it is clear that the fluid properties (vorticity frozen into fluid) described by the Kelvin's theorem are very important for calculating rotational flows, whereas for irrotational flows the constraints imposed by the Kelvin's theorem degenerate into identities which are fulfilled automatically.

The conventional theory of fluid flows is based mainly on the Euler system (1.1)-(1.3) which does not take into account constrains of the Kelvin's theorem. Theory of irrotational flows has been developed well enough, whereas that of rotational flows has been developed much slightly. Apparently, it is connected with the fact that the Euler system (1.1)-(1.3) does not contain enough information on the fluid properties, and proper calculations of rotational flows are impossible in the scope of the curtailed Euler system (1.1)-(1.3). Especially it concerns strongly rotational (turbulent) flows. It will be shown in this paper that slightly rotational flows of incompressible fluid can be calculated on the basis of the integrated system (1.4), (1.12), (1.13) (DTHP or DTWF), where integrals (1.11) are taken into account.

The integrals (1.11) can be interpreted as a contour-free form of the Kelvin's theorem in application to the perfect fluid. Indeed, let us multiply (1.11) by $d\mathbf{x}$ and integrate along a closed path $\mathcal{L}$. One obtains

$$C_{\mathcal{L}} \equiv \oint_{\mathcal{L}} \mathbf{v}d\mathbf{x} = \oint_{\mathcal{L}_{\boldsymbol{\xi}}} g^{\alpha}(\boldsymbol{\xi})d\xi_{\alpha} + \oint_{\mathcal{L}_{\boldsymbol{\xi}}} \eta \frac{\partial S_0(\boldsymbol{\xi})}{\partial \xi_{\alpha}} d\xi_{\alpha} \tag{1.18}$$

where $\mathcal{L}_{\boldsymbol{\xi}}$ is a mapping of the contour $\mathcal{L}$ in the $\mathbf{x}$-space onto the $\boldsymbol{\xi}$-space of labels $\boldsymbol{\xi}$. This mapping

$$\mathcal{L} \leftrightarrow \mathcal{L}_{\boldsymbol{\xi}}. \qquad \boldsymbol{\xi} \rightarrow \mathbf{x}(t,\boldsymbol{\xi}), \qquad \mathbf{x} \rightarrow \boldsymbol{\xi}(t,\mathbf{x}) \tag{1.19}$$

depends on time $t$. Let the contour $\mathcal{L}$ be coupled rigidly with fluid particles and move with the fluid. It means that $\mathcal{L}_{\boldsymbol{\xi}}$ is fixed, $\mathbf{x}(t,\boldsymbol{\xi})$ satisfies (1.6), and the shape of $\mathcal{L}$ depends on time according to (1.19), (1.6). Let the flow be homoentropic ($S(\boldsymbol{\xi})$ =const identically), or the contour $\mathcal{L}_{\boldsymbol{\xi}}$ lies on the surface $S(\boldsymbol{\xi})$ =const., the second integral in rhs of (1.18) vanishes, and the circulation $\mathcal{C}_L$ does not depend on time.

Let now the infinitesimal contour $\mathcal{L}$ be a parallelogram made up by two infinitesimal vectors $d\mathbf{x}_1$, $d\mathbf{x}_2$ and $\mathcal{L}_{\boldsymbol{\xi}}$ be made up by two infinitesimal vectors $d\boldsymbol{\xi}_1$, $d\boldsymbol{\xi}_2$. Then using Stokes's theorem, one derives from (1.18)

$$\boldsymbol{\omega}d\mathbf{S} = \boldsymbol{\Omega}(\boldsymbol{\xi})d\mathbf{S}_{\boldsymbol{\xi}}, \qquad \boldsymbol{\omega} = \nabla \times \mathbf{v}, \qquad \boldsymbol{\Omega}(\boldsymbol{\xi}) = \{\Omega_{\alpha}(\boldsymbol{\xi})\}, \qquad \alpha = 1,2,3 \tag{1.20}$$

$$\Omega_{\alpha} = \varepsilon_{\alpha\beta\gamma}\Omega^{\beta\gamma} \qquad \alpha = 1,2,3 \qquad d\mathbf{S} = d\mathbf{x}_1 \times d\mathbf{x}_2, \qquad d\mathbf{S}_{\boldsymbol{\xi}} = d\boldsymbol{\xi}_1 \times d\boldsymbol{\xi}_2$$

$$\Omega^{\alpha\beta} = \frac{\partial g^{\alpha}(\boldsymbol{\xi})}{\partial \xi_{\beta}} - \frac{\partial g^{\beta}(\boldsymbol{\xi})}{\partial \xi_{\alpha}}, \qquad \alpha,\beta = 1,2,3 \tag{1.21}$$

where $\varepsilon_{\alpha\beta\gamma}$ is the Levi-Chivita pseudotensor, and a summation is produced (1-3) over repeated Greek indices. d$\mathbf{S}$ and d$\mathbf{S}_{\xi}$ are infinitesimal area of contours $\mathcal{L}$ and $\mathcal{L}_{\boldsymbol{\xi}}$ respectively. The relation (1.20) is a local (or contour-free) form of the Kelvin's theorem. The scalar $\boldsymbol{\omega}d\mathbf{S}$ conserves and does not depend on time, although $\boldsymbol{\omega}$ and $d\mathbf{S}$ individually depend on time. For any irrotational flow the relation (1.20) degenerates into identity, and Kelvin's theorem may be ignored.

Note that $\boldsymbol{\Omega}(\boldsymbol{\xi})$ may be regarded as a "frozen vorticity", because it depends on time $t$ only via $\boldsymbol{\xi}$, and $\boldsymbol{\Omega} = \boldsymbol{\omega}$, provided $\boldsymbol{\xi} = \mathbf{x}$. $\boldsymbol{\Omega}(\boldsymbol{\xi})$ is a scalar in the $\mathbf{x}$-space, and it is a vector in the $\boldsymbol{\xi}$-space of labels $\boldsymbol{\xi}$.

For derivation of integrated system (1.4), (1.12), (1.13) one uses a specific mathematical technique based on Jacobian properties. This technique permits to integrate dynamic equations without a use of a change of variables. In general, all results can be obtained, carrying out a proper change of variables in the action functional by means of the Lagrange multipliers. Such changes of dependent variables are produced in the Hamilton fluid dynamics (see, for instance, Salmon, 1988). Dynamic equations of the type of (1.3) appear as a result of such changes of variables. These equations can be integrated easily on the basis of equations (1.4). Unfortunately, such changes of variables lead to different sets of dependent variables whose physical meaning is unclear. In other words, using Lagrange multipliers, for a change of variables, one obscures logical connection between different variables. It is rather difficult to understand that, integrating some equations of the type of (1.3), one integrates in reality the equations (1.2). To simplify the logical connection between different dependent variables and to clear their physical meaning, we prefer to integrate dynamic equations directly by means of "Jacobian technique" (sec. 2). Use of Jacobians in hydrodynamics has had a long history, dating back to the time of Clebsch (1857, 1859). It was the use of Jacobians that allowed to introduce the Clebsch potentials and integrate hydrodynamic equations.

The Jacobian technique was used by many authors (Herivel (1955), Eckart(1960), Berdichevski (1983), Salmon (1988), Zacharov and Kuznetsov (1997) and many others). We use space-time symmetric version of the Jacobian technique which appears to be simple and effective. It seems that the progress in the integration of hydrodynamic equations is connected mainly with the developed Jacobian technique.

## 2 Jacobian technique

Let us consider such a space-time symmetric mathematical object as the Jacobian

$$J \equiv \frac{\partial(\xi_0, \xi_1, \xi_2, \xi_3)}{\partial(x^0, x^1, x^2, x^3)} \equiv \det \parallel \xi_{i,k} \parallel, \qquad \xi_{i,k} \equiv \partial_k \xi_i \equiv \frac{\partial \xi_i}{\partial x^k}, \qquad i,k = 0,1,2,3 \tag{2.1}$$

Here $\xi = \{\xi_0, \boldsymbol{\xi}\} = \{\xi_0, \xi_1, \xi_2, \xi_3\}$ are four scalar considered as functions of $x = \{x^0, \mathbf{x}\}$, $\xi = \xi(x)$. The functions $\{\xi_0, \xi_1, \xi_2, \xi_3\}$ are supposed to be independent in the sense that $J \neq 0$. It is useful to consider the Jacobian $J$ as 4-linear function of variables $\xi_{i,k} \equiv \partial_k \xi_i$, $i,k = 0,1,2,3$. Then one can introduce derivatives of $J$ with respect to $\xi_{i,k}$. The derivative $\partial J/\partial \xi_{i,k}$ appears as a result of a substitution of $\xi_i$ by $x^k$ in the relation (2.1).

$$\frac{\partial J}{\partial \xi_{i,k}} \equiv \frac{\partial(\xi_0, ...\xi_{i-1}, x^k, \xi_{i+1}, ...\xi_3)}{\partial(x^0, x^1, x^2, x^3)}, \qquad i,k = 0,1,2,3 \tag{2.2}$$

For instance

$$\frac{\partial J}{\partial \xi_{0,i}} \equiv \frac{\partial(x^i, \xi_1, \xi_2, \xi_3)}{\partial(x^0, x^1, x^2, x^3)}, \qquad i = 0,1,2,3 \tag{2.3}$$

This rule is valid for higher derivatives of $J$ also.

$$\frac{\partial^2 J}{\partial \xi_{i,k} \partial \xi_{s,l}} \equiv \frac{\partial(\xi_0, ...\xi_{i-1}, x^k, \xi_{i+1}, ...\xi_{s-1}, x^l, \xi_{s+1}, ...\xi_3)}{\partial(x^0, x^1, x^2, x^3)} \equiv$$

$$\frac{\partial(x^k, x^l)}{\partial(\xi_i, \xi_s)} \frac{\partial(\xi_0, \xi_1, \xi_2, \xi_3)}{\partial(x^0, x^1, x^2, x^3)} \equiv J\left(\frac{\partial x^k}{\partial \xi_i} \frac{\partial x^l}{\partial \xi_s} - \frac{\partial x^k}{\partial \xi_s} \frac{\partial x^l}{\partial \xi_i}\right), \qquad i,k,l,s = 0,1,2,3 \tag{2.4}$$

It follows from (2.1), (2.2) that

$$\frac{\partial x^k}{\partial \xi_i} \equiv \frac{\partial(\xi_0, ...\xi_{i-1}, x^k, \xi_{i+1}, ...\xi_3)}{\partial(\xi_0, \xi_1, \xi_2, \xi_3)} \equiv \frac{\partial(\xi_0, ...\xi_{i-1}, x^k, \xi_{i+1}, ...\xi_3)}{\partial(x^0, x^1, x^2, x^3)} \times$$

$$\frac{\partial(x^0, x^1, x^2, x^3)}{\partial(\xi_0, \xi_1, \xi_2, \xi_3)} \equiv \frac{1}{J} \frac{\partial J}{\partial \xi_{i,k}}, \qquad i,k = 0,1,2,3 \tag{2.5}$$

and (2.4) may be written in the form

$$\frac{\partial^2 J}{\partial\xi_{i,k}\partial\xi_{s,l}} \equiv \frac{1}{J}(\frac{\partial J}{\partial\xi_{i,k}}\frac{\partial J}{\partial\xi_{s,l}} - \frac{\partial J}{\partial\xi_{i,l}}\frac{\partial J}{\partial\xi_{s,k}}), \qquad i,k,l,s = 0,1,2,3 \tag{2.6}$$

The derivative $\partial J/\partial\xi_{i,k}$ is a cofactor to the element $\xi_{i,k}$ of the determinant (2.1). Then one has the following identities

$$\xi_{l,k}\frac{\partial J}{\partial\xi_{s,k}} \equiv \delta_l^s J, \qquad \xi_{k,l}\frac{\partial J}{\partial\xi_{k,s}} \equiv \delta_l^s J, \qquad l,s = 0,1,2,3 \tag{2.7}$$

$$\partial_k\frac{\partial J}{\partial\xi_{i,k}} \equiv \frac{\partial^2 J}{\partial\xi_{i,k}\partial\xi_{s,l}}\partial_k\partial_l\xi_s \equiv 0, \qquad i = 0,1,2,3. \tag{2.8}$$

Here and further a summation on two repeated indices is produced (0-3) for Latin indices and (1-3) for the Greek ones. The identity (2.8) can be considered as a corollary of the identity (2.6) and a symmetry of $\partial_k\partial_l\xi_s$ with respect to permutation of indices $k$, $l$. Convolution of (2.6) with $\partial_k$, or $\partial_l$ vanishes also.

Relations (2.1) –(2.6) are written for four independent variables $x$, but they are valid in an evident way for arbitrary number $n+1$ of variables $x = \{x^0, x^1, \ldots x^n\}$ and $\boldsymbol{\xi} = \{\xi_0, \boldsymbol{\xi}\}$, $\quad \boldsymbol{\xi} = \{\xi_1, \xi_2, \ldots \xi_n\}$.

Application of the Jacobian $J$ to hydrodynamics is founded on the property, that the fluid flux

$$j^i = m\frac{\partial J}{\partial\xi_{0,i}}, \qquad j = \{j^i\} = \{\rho, \rho\mathbf{v}\}, \qquad i = 0,1,2,3 \tag{2.9}$$

constructed on the basis of the variables $\boldsymbol{\xi} = \{\xi_1, \xi_2, \xi_3\}$ satisfies Lin constraints (1.4) and the continuity equation

$$\partial_i j^i = 0 \tag{2.10}$$

identically for any choice of variables $\boldsymbol{\xi}$, as it follows from the identity (2.8) for $i = 0$. The continuity equation (2.10) is used without approximations in all hydrodynamic models, and the change of variables $\{\rho, \rho\mathbf{v}\} \leftrightarrow \boldsymbol{\xi}$ described by (2.9) is very important.

In particular, in the case of two-dimensional established flow of incompressible fluid the variables $\boldsymbol{\xi}$ reduce to one variable $\xi_1 = \psi$, known as the stream function. In this case there are only two essential dependent variables $x^0 = x$, $x^1 = y$, and the relations (2.9), (2.10) reduce to relations

$$\rho^{-1}j_x = v_x = \frac{\partial\psi}{\partial y}, \qquad \rho^{-1}j_y = v_y = -\frac{\partial\psi}{\partial x}, \qquad \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \tag{2.11}$$

Defining the stream line as a line tangent to the flux $j$

$$\frac{dx}{j_x} = \frac{dy}{j_y}, \tag{2.12}$$

one obtains that the stream function is constant along the stream line, because according to two first equations (2.11), $\psi = \psi(x,y)$ is an integral of the equation (2.12).

In the general case, when the space dimensionality is $n$ and $x = \{x^0, x^1, \ldots x^n\}$, $\xi = \{\xi_0, \boldsymbol{\xi}\}$, $\boldsymbol{\xi} = \{\xi_1, \xi_2, \ldots \xi_n\}$, the quantities $\boldsymbol{\xi} = \{\xi_\alpha\}$, $\alpha = 1, 2, \ldots n$ are constant along the line $\mathcal{L}$ tangent to the flux vector $j = \{j^i\}$, $i = 0, 1, \ldots n$

$$\mathcal{L} : \qquad \frac{dx^i}{d\tau} = j^i(x), \qquad i = 0, 1, \ldots n \tag{2.14}$$

where $\tau$ is a parameter along the line $\mathcal{L}$ which is described parametrically by the equation $x = x(\tau)$. This statement is formulated mathematically in the form

$$\frac{d\xi_\alpha}{d\tau} = j^i \partial_i \xi_\alpha = m \frac{\partial J}{\partial \xi_{0,i}} \partial_i \xi_\alpha = 0, \qquad \alpha = 1, 2, \ldots n \tag{2.15}$$

The last equality follows from the first identity (2.7) taken for $s = 0$, $l = 1, 2, \ldots n$

Interpretation of the line (2.14) tangent to the flux is different for different cases. If $x = \{x^0, x^1, \ldots x^n\}$ contains only spatial coordinates, the line (2.14) is a line in the usual space. It is regarded as a stream line, and $\boldsymbol{\xi}$ can be interpreted as quantities which are constant along the stream line (i.e. as a generalized stream function). If $x^0$ is the time coordinate, the equation (2.14) describes a line in the space-time. This line (known as a world line of a fluid particle) determines a motion of the fluid particle. Variables $\boldsymbol{\xi} = \{\xi_1, \xi_2, \ldots \xi_n\}$ which are constant along the world line are different, generally, for different particles. If $\xi_\alpha$, $\alpha = 1, 2, \ldots n$ are independent, they may be used for the fluid particle labeling.

Thus, although interpretation of the relation (2.9) considered as a change of dependent variables $j$ by $\boldsymbol{\xi}$ may be different, from the mathematical viewpoint this transformation means a replacement of the continuity equation by some equations for the labeling (or generalized stream function) $\boldsymbol{\xi}$. Difference of the interpretation is of no importance in this context.

Note that the expressions

$$j^i = m\rho_0(\boldsymbol{\xi}) \frac{\partial J}{\partial \xi_{0,i}} \equiv m\rho_0(\boldsymbol{\xi}) \frac{\partial(x^i, \xi_1, \xi_2, \xi_3)}{\partial(x^0, x^1, x^2, x^3)}, \qquad i = 0, 1, 2, 3, \tag{2.16}$$

can be also considered as four-flux satisfying the continuity equation (2.10). Here $m$ is a constant and $\rho_0(\boldsymbol{\xi})$ is an arbitrary function of $\boldsymbol{\xi}$. It follows from the identity

$$m\rho_0(\boldsymbol{\xi}) \frac{\partial(x^i, \xi_1, \xi_2, \xi_3)}{\partial(x^0, x^1, x^2, x^3)} \equiv m \frac{\partial(x^i, \tilde{\xi}_1, \xi_2, \xi_3)}{\partial(x^0, x^1, x^2, x^3)}, \qquad \tilde{\xi}_1 = \int_0^{\xi_1} \rho_0(\xi_1', \xi_2, \xi_3) d\xi_1'. \tag{2.17}$$

As an example of application of the Jacobian technique, let us show that (1.10) satisfies (1.1) in virtue of (1.4). Let us multiply (1.4) by (1.10) and introduce new variables $\mathbf{j} = \rho\mathbf{v} = \{j^1, j^2, j^3\}$. One obtains three equations

$$m\rho_0(\boldsymbol{\xi}) \frac{\partial J}{\partial \xi_{0,0}} \xi_{\beta,0} + j^\alpha \xi_{\beta,\alpha} = 0, \qquad \beta = 1, 2, 3. \tag{2.18}$$

Considering (2.18) as a system of three linear equations for $j^\alpha$, $\alpha = 1, 2, 3$ and resolving it with respect to $j^\alpha$, one obtains

$$j^\alpha = m\rho_0(\boldsymbol{\xi}) \frac{\partial J}{\partial \xi_{0,\alpha}}, \qquad \alpha = 1, 2, 3 \tag{2.19}$$

It is easy to verify this, substituting (2.19) into (2.18) and using (2.7). One obtains that $j = \{j^0, \mathbf{j}\} = \{\rho, \rho\mathbf{v}\}$ is described by the relations (2.16) which satisfy the continuity equation (2.10) identically. Thus, (1.1) is satisfied by (1.10) in virtue of (1.4).

# 3 Variational principle

In general, equivalency of the system (1.4), (1.12), (1.13) and the system (1.1)–(1.4) can be verified by a direct substitution of variables $\rho$, $S$, $\mathbf{v}$, defined by the relations (1.9)–(1.11), into the equations (1.1)–(1.3). Using equations (1.4), (1.12), (1.13), one obtains identities after subsequent calculations. But such computations do not display a connection between the integration and the invariancy with respect to the relabeling group (1.16). Besides a meaning of new variables $\varphi$, $\eta$ is not clear. We shall use for our investigations a variational principle. Note that for a long time a derivation of a variational principle for hydrodynamic equaitons (1.1)–(1.3) was existing as a self-dependent problem (Davydov, 1949; Herivel, 1955; Eckart, 1960; Lin, 1963; Seliger and Whithem, 1967; Bretherton, 1970; Salmon, 1988). Existence of this problem was connected with a lack of understanding that the system of hydrodynamic equations (1.1)–(1.3) is a curtailed system, and the full system of dynamic equations (1.1)–(1.4) includes equations (1.4) describing a motion of the fluid particles in the given velocity field. The variational principle can generate only the complete system of dynamic variables (but not its closed subsystem). Without understanding this one tried to form the Lagrangian for the system (1.1)–(1.3) as a sum of some quantities taken with Lagrange multipliers. lhs of dynamic equations (1.1)–(1.3) and some other constraints were taken as such quantities.

Now this problem has been solved (see review by Salmon, 1988) on the basis of the Eulerian version of the variational principle for the Lagrangian description, where equations (1.4) appear automatically and cannot be ignored. In our version of the variational principle we follow Salmon (1988) with some modifications which underline a curtailed character of hydrodynamic equations (1.1)–(1.3), because the understanding of the curtailed character of the system (1.1)–(1.3) removes the problem of derivation of the variational principle for the hydrodynamic equations (1.1)–(1.3).

We consider the ideal fluid as a conservative dynamic system whose dynamic equations can be derived from the variational principle. This dynamic system is a continuous set of many identical particles moving in some self-consistent (and external) potential force field. The action functional has the form

$$\mathcal{A}_{\mathrm{L}}[\mathbf{x}] = \int \{ \frac{m}{2} (\frac{d\mathbf{x}}{dt})^2 - V \} \rho_0(\boldsymbol{\xi}) dt d\boldsymbol{\xi}, \tag{3.1}$$

where $\mathbf{x} = \{x^\alpha(t, \boldsymbol{\xi})\}$, $\alpha = 1, 2, 3$ are dependent variables considered as functions of time $t$ and of labels (Lagrangian coordinates) $\boldsymbol{\xi} = \{\xi_1, \xi_2, \xi_3\}$. $d\mathbf{x}/dt$ is a derivative of $\mathbf{x}$ with respect to $t$ taken at fixed $\boldsymbol{\xi}$.

$$\frac{dx^\alpha}{dt} \equiv \frac{\partial(x^\alpha, \xi_1, \xi_2, \xi_3)}{\partial(t, \xi_1, \xi_2, \xi_3)} \equiv \frac{\partial(x^\alpha, \boldsymbol{\xi})}{\partial(t, \boldsymbol{\xi})} \qquad \alpha = 1, 2, 3 \tag{3.2}$$

$\rho_0(\boldsymbol{\xi})$ is some non-negative weight function, and $V$ is a potential of a self-consistent force field which depends on $\boldsymbol{\xi}$, $\mathbf{x}$ and derivatives of $\mathbf{x}$ with respect to $\boldsymbol{\xi}$. $m$ =const is some mass of the fluid particle. The form of the potential $V$ will be fixed later. Now it is important only that $V$ does not depend on the time derivatives of $\mathbf{x}$.

Variation of the action with respect to $\mathbf{x}$ generates six first order dynamic equations for six dependent variables $\mathbf{x}$, $\mathbf{v} = d\mathbf{x}/dt$, considered as functions of $t$ and of independent curvilinear Lagrangian coordinates $\boldsymbol{\xi}$. It is a Lagrangian representation of hydrodynamic equations.

We prefer to work with Eulerian representation, when Lagrangian coordinates (particle labeling) $\xi = \{\xi_0, \boldsymbol{\xi}\}$, $\boldsymbol{\xi} = \{\xi_1, \xi_2, \xi_3\}$ are considered as dependent variables, and Eulerian coordinates $x = \{x^0, \mathbf{x}\} = \{t, \mathbf{x}\}$, $\mathbf{x} = \{x^1, x^2, x^3\}$ are considered as independent variables. Here $\xi_0$ is a temporal Lagrangian coordinate which evolves along the particle trajectory in an arbitrary way. Now the $\xi_0$ is a fictive variable, but after integration of equations the $\xi_0$ stops to be fictive and turns to the variable $\varphi$, appearing in the integrated system (1.4), (1.12), (1.13).

Further mainly space-time symmetric designations will be used, that simplifies considerably all computations. In the Eulerian description the action functional (3.1) is to be represented as an integral over independent variables $x = \{x^0, \mathbf{x}\} = \{t, \mathbf{x}\}$. One uses the Jacobian technique for such a transformation of the action (3.1),

Let us note that according to (2.3) the derivative (3.2) can be written in the form

$$v^\alpha = \frac{dx^\alpha}{dt} \equiv \frac{\partial J}{\partial \xi_{0,\alpha}} \left( \frac{\partial J}{\partial \xi_{0,0}} \right)^{-1}, \qquad \alpha = 1, 2, 3. \tag{3.3}$$

Then components of the 4-flux $j = \{j^0, \mathbf{j}\} \equiv \{\rho, \rho\mathbf{v}\}$ can be written in the form (2.16), provided the designation (1.10)

$$j^0 = \rho = m\rho_0(\boldsymbol{\xi}) \frac{\partial J}{\partial \xi_{0,0}} \equiv m\rho_0(\boldsymbol{\xi}) \frac{\partial(x^0, \xi_1, \xi_2, \xi_3)}{\partial(x^0, x^1, x^2, x^3)} \tag{3.4}$$

is used.

At such form of the mass density $\rho$ the four-flux $j = \{j^i\}$, $i = 0, 1, 2, 3$ satisfies identically the continuity equation (2.10) which takes place in virtue of identities (2.7), (2.8). Besides in virtue of identities (2.7), (2.8) the Lin constraints (1.4) are fulfilled identically

$$j^i \partial_i \xi_\alpha = 0, \qquad \alpha = 1, 2, 3. \tag{3.5}$$

Components $j^i$ are invariant with respect to the relabeling group (1.16), provided the function $\rho_0(\boldsymbol{\xi})$ transforms as follows

$$\rho_0(\boldsymbol{\xi}) \to \tilde{\rho}_0(\tilde{\boldsymbol{\xi}}) = D^{-1}\rho_0(\boldsymbol{\xi}), \qquad D = \frac{\partial(\tilde{\boldsymbol{\xi}})}{\partial(\boldsymbol{\xi})} \equiv \frac{\partial(\tilde{\xi}_1, \tilde{\xi}_2, \tilde{\xi}_3)}{\partial(\xi_1, \xi_2, \xi_3)} \tag{3.6}$$

One has

$$\rho_0(\boldsymbol{\xi}) dt d\boldsymbol{\xi} = \rho_0(\boldsymbol{\xi}) \frac{\partial J}{\partial \xi_{0,0}} dt d\mathbf{x} = \frac{\rho}{m} dt d\mathbf{x} \tag{3.7}$$

$$\frac{m}{2} \left( \frac{dx^\alpha}{dt} \right)^2 = \frac{m}{2} \left( \frac{\partial J}{\partial \xi_{0,\alpha}} \right)^2 \left( \frac{\partial J}{\partial \xi_{0,0}} \right)^{-2} = \frac{m}{2} \left( \frac{j^\alpha}{\rho} \right)^2, \tag{3.8}$$

and the variational problem with the action functional (3.1) is written as a variational problem with the action functional

$$\mathcal{A}_{\mathrm{E}}[\boldsymbol{\xi}] = \int (\frac{\mathbf{j}^2}{2\rho} - \rho E) dt d\mathbf{x}, \qquad E = \frac{V}{m} \tag{3.9}$$

where $\rho = j^0$ and $\mathbf{j} = \{j^1, j^2, j^3\}$ are fixed functions of $\xi = \{\xi_0, \boldsymbol{\xi}\}$ and of $\xi_{\alpha,i} \equiv \partial_i \xi_\alpha$, $\alpha = 1, 2, 3$, $i = 0, 1, 2, 3$, defined by the relations (2.16). $E$ is the internal energy of the fluid which is supposed to be a fixed function of $\rho$ and $S_0(\boldsymbol{\xi})$

$$E = E(\rho, S_0(\boldsymbol{\xi})), \tag{3.10}$$

where $\rho$ is defined by (3.4) and $S_0(\boldsymbol{\xi})$ is some fixed function of $\boldsymbol{\xi}$, describing initial distribution of the entropy over the fluid.

The action (3.9) is invariant with respect to subgroup $\mathcal{G}_{S_0}$ of the relabeling group (1.16). The subgroup $\mathcal{G}_{S_0}$ is determined in such a way that any surface $S_0(\boldsymbol{\xi})$ =const is invariant with respect to $\mathcal{G}_{S_0}$. In general, the subgroup $\mathcal{G}_{S_0}$ is determined by two arbitrary functions of $\boldsymbol{\xi}$.

The action (3.9) generates the six order system of dynamic equations, consisting of three second order equations for three dependent variables $\boldsymbol{\xi}$. Invariancy of the action (3.9) with respect to the subgroup $\mathcal{G}_{S_0}$ admits one to integrate the system of dynamic equations. The order of the system is reduced, and two arbitrary integration functions appear. The order of the system is reduced to five (but not to four), because the fictive dependent variable $\xi_0$ stops to be fictive as a result of the integration.

Unfortunately, the subgroup $\mathcal{G}_{S_0}$ depends on the form of the function $S_0(\boldsymbol{\xi})$ and cannot be obtained in a general form. In the special case, when $S_0(\boldsymbol{\xi})$ does not depend on $\boldsymbol{\xi}$, the subgroup $\mathcal{G}_{S_0}$ coincides with the whole relabeling group $\mathcal{G}$, and the order of the integrated system is reduced to four.

In the general case it is convenient to introduce a new dependent variable

$$S = S_0(\boldsymbol{\xi}). \tag{3.11}$$

According to (3.5) the variable $S$ satisfies the dynamic equation (1.3)

$$j^i \partial_i S = 0. \tag{3.12}$$

In virtue of designations (2.9) and identities (2.7), (2.8) the equations (3.12), (3.5) are fulfilled identically. Hence, they can be added to the action functional (3.9) as side constraints without a change of the variational problem. Adding (3.12) to the Lagrangian of the action (3.9) by means of a Lagrange multiplier $\eta$, one obtains

$$\mathcal{A}_{\mathrm{E}}[\boldsymbol{\xi}, \eta, S] = \int \{\frac{\mathbf{j}^2}{2\rho} - \rho E + \eta j^k \partial_k S\} dt d\mathbf{x} \tag{3.13}$$

where the quantities $j = \{\rho, \mathbf{j}\}$ are determined by (2.16), and $E = E(\rho, S)$. The action (3.13) is invariant with respect to the relabeling group $\mathcal{G}$ which is determined by three arbitrary functions of $\boldsymbol{\xi}$. Three arbitrary functions of $\boldsymbol{\xi}$ appear in consequence of the integration of dynamic equations. The integrated system contains five first order equations for dependent variables $\xi_0$, $\boldsymbol{\xi}$, $\eta$. The dependent variable $S$ is substituted by arbitrary indefinite function $S_0(\boldsymbol{\xi})$.

To obtain the dynamic equations, it is convenient to introduce new dependent variables $j^i$, defined by (2.16). Let us introduce the new variables $j^i$ by means of designations (2.16) taken with the Lagrange multipliers $p_i$, $i = 0,1,2,3$. Then the action (3.13) takes the form

$$\mathcal{A}_{\mathrm{E}}[\rho,\mathbf{j},\boldsymbol{\xi},p,\eta,S] = \int\{\frac{\mathbf{j}^2}{2\rho} - \rho E - p_k[j^k - m\rho_0(\boldsymbol{\xi})\frac{\partial J}{\partial \xi_{0,k}}] + \eta j^k\partial_k S\}dtd\mathbf{x} \qquad (3.14)$$

It is useful to keep in mind that four designations (2.16), introducing variables $\rho$, $\mathbf{j} = \rho\mathbf{v}$ via variables $\boldsymbol{\xi}$, are equivalent to three Lin constraints (1.4) together with the designation (3.4), as it was shown in the end of sec.2. Addition of relations (2.16) to the action (3.13) as side constraints is equivalent to the addition of relations (1.4), (3.4) considered as side constraints.

For obtaining dynamic equations the variables $\rho,\mathbf{j},\boldsymbol{\xi},p,\eta,S$ are to be varied. Let us eliminate the variables $p_i$ from the action (3.14). Dynamic equations arising as a result of a variation with respect to $\xi_\alpha$ have the form

$$\frac{\delta\mathcal{A}_{\mathrm{E}}}{\delta\xi_\alpha} \equiv \hat{\mathcal{L}}_\alpha p = -m\partial_k[\rho_0(\boldsymbol{\xi})\frac{\partial^2 J}{\partial\xi_{0,i}\partial\xi_{\alpha,k}}p_i] + m\frac{\partial\rho_0(\boldsymbol{\xi})}{\partial\xi_\alpha}\frac{\partial J}{\partial\xi_{0,k}}p_k = 0, \qquad \alpha = 1,2,3 \qquad (3.15)$$

where $\hat{\mathcal{L}}_\alpha$ are linear operators acting on variables $p = \{p_i\}$, $i = 0,1,2,3$. This equations can be integrated in the form

$$p_i = g^0(\xi_0)\partial_i\xi_0 + g^\alpha(\boldsymbol{\xi})\partial_i\xi_\alpha, \qquad i = 0,1,2,3, \qquad (3.16)$$

where $\xi_0$ is some new variable (temporal Lagrangian coordinate), $g^\alpha(\boldsymbol{\xi})$, $\alpha = 1,2,3$ are arbitrary functions of the label $\boldsymbol{\xi}$, $g^0(\xi_0)$ is an arbitrary function of $\xi_0$. The relations (3.16) satisfy equations (3.15) identically. Indeed, substituting (3.16) into (3.15) and using identities (2.6), (2.7), one obtains

$$-m\partial_k\left\{\rho_0(\boldsymbol{\xi})\left[\frac{\partial J}{\partial\xi_{\alpha,k}}g^0(\xi_0) - \frac{\partial J}{\partial\xi_{0,k}}g^\alpha(\boldsymbol{\xi})\right]\right\} + m\frac{\partial\rho_0(\boldsymbol{\xi})}{\partial\xi_\alpha}Jg^0(\xi_0) = 0, \qquad \alpha = 1,2,3, \qquad (3.17)$$

Differentiating braces and using identities (2.8), (2.7), one concludes that (3.17) is an identity.

Setting for simplicity

$$\partial_k\varphi = g^0(\xi_0)\partial_k\xi_0, \qquad k = 0,1,2,3 \qquad (3.18)$$

one obtains

$$p_k = \partial_k\varphi + g^\alpha(\boldsymbol{\xi})\partial_k\xi_\alpha, \qquad k = 0,1,2,3 \qquad (3.19)$$

Substituting (3.19) in (3.14), one can eliminate variables $p_i$, $i = 0,1,2,3$ from the functional (3.14). The term $g^\alpha(\boldsymbol{\xi})\partial_k\xi_\alpha\partial J/\partial\xi_{0,k}$ vanish, the term $\partial_k\varphi\partial J/\partial\xi_{0,k}$

gives no contribution into dynamic equations. The action functional takes the form

$$\mathcal{A}_{\mathbf{g}}[\rho,\mathbf{j},\boldsymbol{\xi},\eta,S]=\int\{\frac{\mathbf{j}^2}{2\rho}-\rho E-j^k[\partial_k\varphi+g^\alpha(\boldsymbol{\xi})\partial_k\xi_\alpha-\eta\partial_k S]\}dtd\mathbf{x} \tag{3.20}$$

where $g^\alpha(\boldsymbol{\xi})$ are considered as fixed functions of $\boldsymbol{\xi}$ which are determined from initial conditions. Varying the action (3.20) with respect to $\varphi$, $\boldsymbol{\xi}$, $\eta$, $S$, $\mathbf{j}$, $\rho$, one obtains dynamic equations

$$\delta\varphi: \qquad \partial_k j^k=0, \tag{3.21}$$

$$\delta\xi_\alpha: \qquad \Omega^{\alpha\beta}j^k\partial_k\xi_\beta=0, \qquad \alpha=1,2,3, \tag{3.22}$$

where $\Omega^{\alpha\beta}$ is defined by (1.21)

$$\delta\eta: \qquad j^k\partial_k S=0, \tag{3.23}$$

$$\delta S: \qquad j^k\partial_k\eta=-\rho\frac{\partial E}{\partial S}, \tag{3.24}$$

$$\delta\mathbf{j}: \qquad \mathbf{v}\equiv\mathbf{j}/\rho=\nabla\varphi+g^\alpha(\boldsymbol{\xi})\nabla\xi_\alpha-\eta\nabla S, \tag{3.25}$$

$$\delta\rho: \qquad -\frac{\mathbf{j}^2}{2\rho^2}-\frac{\partial(\rho E)}{\partial\rho}-\partial_0\varphi-g^\alpha(\boldsymbol{\xi})\partial_0\xi_\alpha+\eta\partial_0 S=0, \tag{3.26}$$

Deriving relations (3.22), (3.24), the continuity equation (3.21) was used. It is easy to see that (3.22) is equivalent to (1.4), provided

$$\det\parallel\Omega^{\alpha\beta}\parallel\neq 0 \tag{3.27}$$

Then the equations (3.23) and (3.21) can be integrated in the form of (1.9) and (1.10) respectively. Equations (3.24) and (3.25) are equivalent to (1.13) and (1.11). Finally, eliminating $\partial_0\xi_\alpha$ and $\partial_0 S$ from (3.26) by means of (3.22) and (3.23), one obtains the equation (1.12) and, hence, the system of dynamic equations (1.4), (1.12), (1.13), where designations (1.9)–(1.11) are used.

The curtailed system (1.1)–(1.3) can be obtained from equations (3.21)–(3.26) as follows. Equations (3.21), (3.23) coincide with (1.1), (1.3). For deriving (1.2) let us note that the vorticity $\boldsymbol{\omega}\equiv\nabla\times\mathbf{v}$ and $\mathbf{v}\times\boldsymbol{\omega}$ are obtained from (3.25) in the form

$$\boldsymbol{\omega}=\nabla\times\mathbf{v}=\frac{1}{2}\Omega^{\alpha\beta}\nabla\xi_\beta\times\nabla\xi_\alpha-\nabla\eta\times\nabla S \tag{3.28}$$

$$\mathbf{v}\times\boldsymbol{\omega}=\Omega^{\alpha\beta}\nabla\xi_\beta(\mathbf{v}\nabla)\xi_\alpha+\nabla S(\mathbf{v}\nabla)\eta-\nabla\eta(\mathbf{v}\nabla)S \tag{3.29}$$

Let us form a difference between the time derivative of (3.25) and the gradient of (3.26). Eliminating $\Omega^{\alpha\beta}\partial_0\boldsymbol{\xi}_\alpha$, $\partial_0 S$ and $\partial_0\eta$ by means of equations (3.22), (3.23), (3.24), one obtains

$$\partial_0\mathbf{v}+\nabla\frac{\mathbf{v}^2}{2}+\frac{\partial^2(\rho E)}{\partial\rho^2}\nabla\rho+\frac{\partial^2(\rho E)}{\partial\rho\partial S}\nabla S-\rho\frac{\partial E}{\partial S}\nabla S$$

$$-\Omega^{\alpha\beta}\nabla\xi_\beta(\mathbf{v}\nabla)\xi_\alpha+\nabla\eta(\mathbf{v}\nabla)S-\nabla S(\mathbf{v}\nabla)\eta=0 \tag{3.30}$$

Using (3.28), (3.29) the expression (3.30) reduces to

$$\partial_0\mathbf{v}+\nabla\frac{\mathbf{v}^2}{2}+\frac{1}{\rho}\nabla(\rho^2\frac{\partial E}{\partial\rho})-\mathbf{v}\times(\nabla\times\mathbf{v})=0 \tag{3.31}$$

In virtue of the identity

$$\mathbf{v}\times(\nabla\times\mathbf{v})\equiv\nabla\frac{\mathbf{v}^2}{2}-(\mathbf{v}\nabla)\mathbf{v} \tag{3.32}$$

the last equation is equivalent to (1.2).

Thus, differentiating equations (3.25), (3.26) and eliminating the variables $\varphi$, $\boldsymbol{\xi}$, $\eta$, one obtains the curtailed system (1.1)–(1.3), whereas the system (1.4), (1.12), (1.13) follows from the system (3.21)–(3.26) directly (i.e. without differentiating). It means that the system (1.4), (1.12), (1.13) is an integrated system, whereas the curtailed system (1.1)–(1.3) is not, although formally they have the same order.

The action of the form (3.20), or close to this form was obtained by some authors (Seliger and Whithem, 1967; Salmon, 1988), but the quantities $g^\alpha$, $\alpha=1,2,3$ are always considered as additional dependent variables (but not as indefinite functions of $\boldsymbol{\xi}$ which can be expressed via initial conditions). The action was not considered as a functional of fixed indefinite functions $g^\alpha(\boldsymbol{\xi})$.

The variable $\eta$ was introduced, for the action be invariant with respect to the transformations of the whole relabeling group (1.16). To understand what the $\eta$ means from the mathematical viewpoint, let us return to the action (3.9), where the internal energy $E$ has the form (3.10). Adding new variables $j$ by means of designations (2.16), one obtains instead of (3.14)

$$\mathcal{A}_{\mathrm{E}}[\rho,\mathbf{j},\boldsymbol{\xi},p]=\int\{\frac{\mathbf{j}^2}{2\rho}-\rho E-p_k[j^k-m\rho_0(\boldsymbol{\xi})\frac{\partial J}{\partial\xi_{0,k}}]\}dtd\mathbf{x} \tag{3.33}$$

where $E$ has the form (3.10).

Variation of (3.33) with respect to $\xi_\alpha$ leads to the equation

$$\hat{\mathcal{L}}_\alpha p=\rho\frac{\partial E(\rho,S_0(\boldsymbol{\xi}))}{\partial S_0}\frac{\partial S_0}{\partial\xi_\alpha},\qquad\alpha=1,2,3 \tag{3.34}$$

where linear operators $\hat{\mathcal{L}}_\alpha$ are defined by (3.15). Equations (3.34) are linear non-uniform equations for the variables $p$. A solution of (3.34) is a sum of the general solution (3.19) of the uniform equations (3.15) and of a particular solution the non-uniform equations (3.34). This particular solution depends on the form of the function $S_0$ and cannot be found in a general form. Adding an extraterm $-\eta j^k\partial_kS$ with $\eta$ satisfying (3.24) to (3.13), a reduction of non-uniform equations (3.34) to uniform equations (3.15) appears to be possible. Thus, the extravariable $\eta$ is responsible for the particular solution of (3.34).

From the viewpoint of the action (3.33) a dependence of the internal energy $E$ on the entropy means simply a dependence of $E$ on the labels $\boldsymbol{\xi}$ via a function $S(\boldsymbol{\xi})$. If such a dependence cannot be expressed through one function (for instance

$E = E[\rho, S_1(\boldsymbol{\xi}), S_2(\boldsymbol{\xi})]$) the ideal fluid is described by two entropies $S_1$ and $S_2$ and by two temperatures $T_1 = \partial E/\partial S_1, \quad T_2 = \partial E/\partial S_2$. Such a situation may appear for a conducting fluid in a strong magnetic field, where there are two temperatures – longitudinal and transversal.

Thus five equations (1.4), (1.12), (1.13) with $S$, $\rho$ and $\mathbf{v}$, defined respectively by (1.9), (1.10) and (1.11), constitute the fifth order system for five dependent variables $\xi = \{\xi_0, \boldsymbol{\xi}\}$, $\eta$. Equations (1.1), (1.3),(1.4), (1.12),(1.13) constitute the seventh order system for seven variables $\rho$, $\boldsymbol{\xi}$, $\varphi$, $\eta$, $S$.

# 4 Initial and Boundary Conditions

Boundary conditions describing vessel walls can be taken into account by means of a proper choice of the internal energy $E(x, \rho, S)$ which can include the energy of the fluid in an external potential $U$.

$$E = E_0(\rho, S) + U(t, \mathbf{x}), \tag{4.1}$$

where $U$ is some given external potential. For instance, let the fluid move inside a volume $\mathcal{V}$. Then

$$U(\mathbf{x}) = \begin{cases} 0, & \text{inside} \quad \mathcal{V} \\ \infty, & \text{outside} \quad \mathcal{V} \end{cases}$$

Such a choice of the energy $E$ provides that the fluid does not escape the volume $\mathcal{V}$.

Let us consider the case, when the fluid flow is considered in the space-time region $\Omega$ defined by inequalities

$$\Omega : \qquad t \geq 0, \qquad x^3 \geq 0 \tag{4.2}$$

The region $\Omega$ has two boundaries: $\mathcal{I}$ defined by the relations $t = 0$, $x^3 \geq 0$, and $\mathcal{B}$ defined by the relations $x^3 = 0$, $t \geq 0$. The initial conditions for the system of equations (1.1)–(1.4) have the form

$$\rho(0, \mathbf{x}) = \rho_{\text{in}}(\mathbf{x}), \qquad v^\alpha(0, \mathbf{x}) = v^\alpha_{\text{in}}(\mathbf{x}), \qquad \alpha = 1, 2, 3 \tag{4.3}$$

$$S(0, \mathbf{x}) = S_{\text{in}}(\mathbf{x}), \qquad \xi_\alpha(0, \mathbf{x}) = \xi^\alpha_{\text{in}}(\mathbf{x}), \qquad \alpha = 1, 2, 3 \tag{4.4}$$

at $\mathbf{x} \in \mathcal{I}$ ($t = 0$, $x^3 \geq 0$). Here $\rho_{\text{in}}$, $\mathbf{v}_{\text{in}}$, $S_{\text{in}}$, $\boldsymbol{\xi}_{\text{in}}$ are given functions of argument $\mathbf{x}$. The boundary conditions on the boundary $\mathcal{B}$ of $\Omega$ have the form:

$$\rho(x)|_{x^3=0} = \rho_{\text{b}}(t, \mathbf{y}), \qquad S(x)|_{x^3=0} = S_{\text{b}}(t, \mathbf{y}), \qquad \{t, \mathbf{y}\} \in \mathcal{B} \tag{4.5}$$

$$v^\alpha(x)|_{x^3=0} = v^\alpha_{\text{b}}(t, \mathbf{y}), \qquad \alpha = 1, 2, 3, \qquad \{t, \mathbf{y}\} \in \mathcal{B} \tag{4.6}$$

$$\xi_\alpha(x)|_{x^3=0} = \xi^\alpha_{\text{b}}(t, \mathbf{y}), \qquad \alpha = 1, 2, 3, \qquad \{t, \mathbf{y}\} \in \mathcal{B} \tag{4.7}$$

where

$$\mathbf{y} \equiv \{x^1, x^2\} \tag{4.8}$$

Here $\rho_{\text{b}}$, $S_{\text{b}}$, $\mathbf{v}_{\text{b}}$, $\boldsymbol{\xi}_{\text{b}}$ are given functions of the argument $\{t, \mathbf{y}\}$.

Let us show that indefinite functions $\mathbf{g}$, $S_0$, $\rho_0$ can be expressed via initial and boundary conditions (4.3)–(4.7). The initial conditions for the system (3.21)–(3.26) have the form

$$\xi_\alpha(0,\mathbf{x}) = \xi_{\text{in}}^\alpha(\mathbf{x}), \qquad \alpha = 1,2,3 \tag{4.9}$$

$$\rho(0,\mathbf{x}) = \rho_{\text{in}}(\mathbf{x}), \qquad S(0,\mathbf{x}) = S_0[\boldsymbol{\xi}_{\text{in}}(\mathbf{x})], \tag{4.10}$$

$$\varphi(0,\mathbf{x}) = \varphi_{\text{in}}(\mathbf{x}), \qquad \eta(0,\mathbf{x}) = \eta_{\text{in}}(\mathbf{x}), \tag{4.11}$$

(4.9)-(4.11) take place at $\mathbf{x} \in \mathcal{I}$. The functions $\varphi_{\text{in}}(\mathbf{x}), \eta_{\text{in}}(\mathbf{x})$ as well $g^\alpha(\boldsymbol{\xi})$ are to be determined from the relations

$$\partial_\alpha \varphi_{\text{in}}(\mathbf{x}) + g^\beta[\boldsymbol{\xi}_{\text{in}}(\mathbf{x})]\partial_\alpha \xi_{\text{in}}^\beta(\mathbf{x}) - \eta_{\text{in}}(\mathbf{x})\frac{\partial S_0[\boldsymbol{\xi}_{\text{in}}(\mathbf{x})]}{\partial \xi_{\text{in}}^\beta}\partial_\alpha \xi_{\text{in}}^\beta(\mathbf{x}) =$$

$$= v_{\text{in}}^\alpha(\mathbf{x}), \qquad \alpha = 1,2,3; \qquad \mathbf{x} \in \mathcal{I}. \tag{4.12}$$

It is clear that five functions $\mathbf{g}, \varphi_{\text{in}}, \eta_{\text{in}}$ cannot be determined unambiguously from three relations (4.12).

There are at least two different approaches to determination of functions $\boldsymbol{\xi}_{\text{in}}(\mathbf{x})$ and $\mathbf{g}(\boldsymbol{\xi})$.

(1) One fixes the functions $\xi_{\text{in}}^\alpha(\mathbf{x})$ in some conventional way, sets

$$\varphi_{\text{in}}(\mathbf{x}) = 0, \qquad \eta_{\text{in}}(\mathbf{x}) = 0, \qquad \mathbf{x} \in \mathcal{I} \tag{4.13}$$

and determines functions $\mathbf{g}$ from three relations (4.12).

(2) Functions $\mathbf{g}$ are fixed in some conventional way, and remaining functions are determined from relations (4.12)

*The first way.* Let the condition (4.9) be given in the form

$$\xi_\alpha(0,\mathbf{x}) = \xi_{\text{in}}^\alpha(\mathbf{x}) = x^\alpha, \qquad \alpha = 1,2,3, \qquad \mathbf{x} \in \mathcal{I}. \tag{4.14}$$

In other words, at $t = 0$ the labels $\boldsymbol{\xi}$ coincide with the Eulerian coordinates to within a constant factor. The relations (4.12) take the form

$$g^\beta[\boldsymbol{\xi}_{\text{in}}(\mathbf{x})] = v_{\text{in}}^\beta(\mathbf{x}), \qquad \alpha = 1,2,3; \qquad \mathbf{x} \in \mathcal{I}, \tag{4.15}$$

which are resolved in the form

$$g^\alpha(\boldsymbol{\xi}) = v_{\text{in}}^\alpha(\boldsymbol{\xi}), \qquad \alpha = 1,2,3, \qquad \xi_3 > 0, \tag{4.16}$$

Thus, the functions $\mathbf{g}$ are expressed through initial conditions (4.3).

The boundary conditions for the system of equations (3.21)-(3.26) have the form

$$\xi_\alpha(x)|_{x^3=0} = \xi_{\text{b}}^\alpha(t,\mathbf{y}), \qquad \alpha = 1,2,3, \qquad \{t,\mathbf{y}\} \in \mathcal{B} \tag{4.17}$$

$$S(x)|_{x^3=0} = S_0[\boldsymbol{\xi}_{\text{b}}(t,\mathbf{y})] = S_{\text{b}}(t,\mathbf{y}), \qquad \{t,\mathbf{y}\} \in \mathcal{B}, \tag{4.18}$$

$$\rho(x)|_{x^3=0} = \rho_{\text{b}}(x)|_{x^3=0}, \qquad \mathbf{v}(x)|_{x^3=0} = \mathbf{v}_{\text{b}}(t,\mathbf{y}), \qquad \{t,\mathbf{y}\} \in \mathcal{B}, \tag{4.19}$$

$$\varphi(x)|_{x^3=0} = \eta(x)|_{x^3=0} = 0, \qquad \{t,\mathbf{y}\} \in \mathcal{B}, \tag{4.20}$$

Let us set

$$\xi_{\rm b}^{\alpha}(t,\mathbf{y}) = x^{\alpha}, \qquad \alpha = 1,2; \qquad \xi_{\rm b}^{3}(t,\mathbf{y}) = -ct, \qquad (t,\mathbf{y}) \in \mathcal{B}, \tag{4.21}$$

where $c$ is a constant.

Writing relations (1.4) and (3.26) for $\xi_3 < 0$ on the boundary $\mathcal{B}$ and using (4.20), (4.21), one obtains constraints for the functions $\mathbf{g}(\boldsymbol{\xi})$

$$g^{\beta}[\boldsymbol{\xi}_{\rm b}(t,\mathbf{y})]\partial_{\alpha}\xi_{\rm b}^{\beta}(t,\mathbf{y}) = v_{\rm b}^{\alpha}(t,\mathbf{y}), \qquad \alpha = 1,2, \qquad \{t,\mathbf{y}\} \in \mathcal{B} \tag{4.22}$$

$$g^{\beta}[\boldsymbol{\xi}_{\rm b}(t,\mathbf{y})]\partial_{0}\xi_{\rm b}^{\beta}(t,\mathbf{y}) = -K_{\rm b}(t,\mathbf{y}), \qquad \{t,\mathbf{y}\} \in \mathcal{B}, \tag{4.23}$$

where

$$K_{\rm b}(t,\mathbf{y}) \equiv \frac{\mathbf{v}_{\rm b}^{2}(t,\mathbf{y})}{2} + \frac{\partial\{\rho_{\rm b}(t,\mathbf{y})E[\rho_{\rm b}(t,\mathbf{y}),S_{\rm b}(t,\mathbf{y})]\}}{\partial\rho_{\rm b}(t,\mathbf{y})}, \qquad \{t,\mathbf{y}\} \in \mathcal{B}, \tag{4.24}$$

Substituting relations (4.21) into (4.22), (4.23), one obtains three equations for determination of functions $\mathbf{g}(\boldsymbol{\xi})$. Resolving this system of equations with respect to $\mathbf{g}$, one obtains

$$g^{\alpha}(\boldsymbol{\xi}) = v_{\rm b}^{\alpha}(-\xi_3/c,\xi_1,\xi_2), \qquad \alpha = 1,2; \qquad \xi_3 < 0$$

$$g^{3}(\boldsymbol{\xi}) = c^{-1}K_{\rm b}(-\xi_3/c,\xi_1,\xi_2), \qquad \xi_3 < 0 \tag{4.25}$$

Thus, $\mathbf{g}(\boldsymbol{\xi})$ is determined by (4.16) for $\xi_3 > 0$ and by (4.25) for $\xi_3 < 0$. In other words, the boundary conditions and the initial conditions determine the vector field $\mathbf{g}(\boldsymbol{\xi})$ in different regions of the argument $\boldsymbol{\xi}$. All information about the velocities at the initial moment and on the boundary has been transferred into dynamic equations. The field $\mathbf{g}(\boldsymbol{\xi})$ can describe both initial and boundary conditions.

*The second way.* Let us choose the functions $\mathbf{g}$ in a simple form. Let for instance,

$$g^{1}(\boldsymbol{\xi}) = \xi_2, \qquad g^{2}(\boldsymbol{\xi}) = 0, \qquad g^{3}(\boldsymbol{\xi}) = 0 \tag{4.26}$$

Let us set

$$\chi = \varphi, \qquad \lambda = \xi_2, \qquad \mu = \xi_1 \tag{4.27}$$

Then the expression (1.11) takes the form

$$\mathbf{u}(\chi,\lambda,\mu,\eta,S) \equiv \nabla\chi + \lambda\nabla\mu - \eta\nabla S = \mathbf{v} \tag{4.28}$$

where $\chi,\lambda,\mu,$ are Clebsch potentials (Clebsch, 1857; 1859). Now six equations (1.1), (1.3), (3.22)-(3.26), (3.27) [(3.22) for $\alpha = 3$ is of no importance] for six dependent variables $\rho,\chi,\lambda,\mu,\eta,S$ do not contain indefinite functions and have an unambiguous form.

$$\partial_0\rho + \nabla(\rho\mathbf{u}) = 0, \qquad \partial_0\lambda + (\mathbf{u}\nabla)\lambda = 0$$

$$\partial_0\mu + (\mathbf{u}\nabla)\mu = 0, \qquad \partial_0 S + (\mathbf{u}\nabla)S = 0 \tag{4.29}$$

$$\partial_0\eta + (\mathbf{u}\nabla)\eta = -\frac{\partial E}{\partial S}, \qquad \partial_0\chi + \lambda\partial_0\mu - \eta\partial_0 S + \frac{1}{2}\mathbf{u}^2 + \frac{\partial(\rho E)}{\partial\rho} = 0$$

where $\mathbf{u}$ is defined by (4.28).

The initial conditions for variables $\rho, \chi, \lambda, \mu, \eta, S$ are determined by relations

$$\rho(0, \mathbf{x}) = \rho_{\rm in}(0, \mathbf{x}), \qquad S(0, \mathbf{x}) = S_{\rm in}(0, \mathbf{x}), \tag{4.30}$$

$$\nabla \chi_{\rm in} + \lambda_{\rm in} \nabla \mu_{\rm in} - \eta_{\rm in} \nabla S_{\rm in} = \mathbf{v}_{\rm in} \tag{4.31}$$

Three equations (4.30), (4.31) do not determine the initial conditions

$$\chi(0, \mathbf{x}) = \chi_{\rm in}(\mathbf{x}), \qquad \lambda(0, \mathbf{x}) = \lambda_{\rm in}(\mathbf{x}), \tag{4.32}$$

$$\mu(0, \mathbf{x}) = \mu_{\rm in}(\mathbf{x}), \qquad \eta(0, \mathbf{x}) = \eta_{\rm in}(\mathbf{x}), \tag{4.33}$$

unambiguously.

If the fluid is described in terms of Clebsch potentials, the dynamic equations contain neither arbitrary functions, nor information about the initial conditions. It should be interpreted in the sense that the description (4.28)-(4.29) in terms of the Clebsch potentials is a result of a change of variables in dynamic equations (1.1)-(1.3), whereas the description (3.21)-(3.26) is a result of integration of the dynamic equations (1.1)-(1.4). In other words, the description (4.28)-(4.29) in terms of Clebsch potentials relates to the description (3.21)-(3.26) in the same way, as a particular solution of a system of differential equations relates to a general solution of the same system.

Let us note that there are many other ways for determination of indefinite functions $\mathbf{g}(\boldsymbol{\xi})$. For instance, for slightly rotational flows the functions $\mathbf{g}(\boldsymbol{\xi})$ may be chosen as small corrections to the basic irrotational flow described by the potential $\varphi$.

# 5 Description in Lagrangian coordinates

To show that the system (1.4), (1.9)-(1.13) is indeed the integrated system of hydrodynamic equations, let us rewrite it in Lagrangian coordinates, when five variables $\mathbf{x} = \{x^1, x^2, x^3\}, \varphi, \eta$ are considered as functions of four independent variables $t, \boldsymbol{\xi} = \{\xi_1, \xi_2, \xi_3, \}$. It is necessary to introduce designations

$$Q \equiv \frac{\partial(x^1, x^2, x^3)}{\partial(\xi_1, \xi_2, \xi_3)} \equiv \frac{\partial(\mathbf{x})}{\partial(\boldsymbol{\xi})} \equiv \det \parallel x^{\alpha,\beta} \parallel, \qquad x^{\alpha,\beta} \equiv \frac{\partial x^\alpha}{\partial \xi_\beta}, \qquad \alpha, \beta = 1, 2, 3 \tag{5.1}$$

$$X_{\alpha,\beta} \equiv \frac{\partial Q}{\partial x^{\alpha,\beta}}, \qquad \alpha, \beta = 1, 2, 3 \tag{5.2}$$

It follows from (2.3), (3.4) that

$$Q = (\frac{\partial J}{\partial \xi_{0,0}})^{-1} = \frac{m\rho_0(\boldsymbol{\xi})}{\rho}. \tag{5.3}$$

Identities (2.7), (2.8) take the form

$$x^{\alpha,\beta} X_{\gamma,\beta} \equiv \delta^\alpha_\gamma Q, \qquad \frac{\partial}{\partial \xi_\gamma} X_{\alpha,\gamma} \equiv 0, \qquad \alpha, \beta = 1, 2, 3. \tag{5.4}$$

Derivative with respect $x^1$ can be recalculated into derivative with respect to $\xi_\alpha$ as follows

$$\frac{\partial\varphi}{\partial x^1} \equiv \frac{\partial(\varphi, x^2, x^3)}{\partial(x^1, x^2, x^3)} \equiv \frac{\partial(\varphi, x^2, x^3)}{\partial(\xi_1, \xi_2, \xi_3)}\frac{\partial(\xi_1, \xi_2, \xi_3)}{\partial(x^1, x^2, x^3)} = Q^{-1}X_{1,\beta}\frac{\partial\varphi}{\partial\xi_\beta}$$

or for a derivative with respect to $x^\alpha$

$$\frac{\partial\varphi}{\partial x^\alpha} \equiv Q^{-1}X_{\alpha,\beta}\frac{\partial\varphi}{\partial\xi_\beta} \qquad \alpha = 1, 2, 3 \tag{5.6}$$

Applying the rule (5.6) to (1.11), one obtains

$$v^\alpha = v^\alpha(t, \boldsymbol{\xi}) = Q^{-1}X_{\alpha,\beta}[\frac{\partial\varphi}{\partial\xi_\beta} + g^\beta(\boldsymbol{\xi}) - \eta\frac{\partial S_0(\boldsymbol{\xi})}{\partial\xi_\beta}], \qquad \alpha = 1, 2, 3 \tag{5.7}$$

Let $D/Dt$ means derivative with respect to $t$ at constant $\boldsymbol{\xi}$

$$\frac{D\varphi}{Dt} \equiv \partial_0\varphi + \frac{Dx^\alpha}{Dt}\partial_\alpha\varphi \tag{5.8}$$

In particular

$$\frac{D\xi_\alpha}{Dt} = \partial_0\xi_\alpha + \frac{Dx^\beta}{Dt}\partial_\beta\xi_\alpha = 0, \tag{5.9}$$

because $D\xi_\alpha/Dt = 0$ by definition. Comparing (5.9) with (1.4) and using (5.7), one concludes that

$$v^\alpha = \frac{Dx^\alpha}{Dt} = Q^{-1}X_{\alpha,\beta}[\frac{\partial\varphi}{\partial\xi_\beta} + g^\beta(\boldsymbol{\xi}) - \eta\frac{\partial S_0(\boldsymbol{\xi})}{\partial\xi_\beta}], \qquad \alpha = 1, 2, 3 \tag{5.10}$$

Now in virtue of (5.6), (5.7), (5.10) the equations (1.12), (1.13) can be rewritten in the form

$$\frac{D\varphi}{Dt} - \frac{1}{2}\sum_{\alpha=1}^{\alpha=3}\left\{Q^{-1}X_{\alpha,\beta}\left[\frac{\partial\varphi}{\partial\xi_\beta} + g^\beta(\boldsymbol{\xi}) - \eta\frac{\partial S_0(\boldsymbol{\xi})}{\partial\xi_\beta}\right]\right\}^2 + \frac{\partial[\rho E(\rho, S)]}{\partial\rho} = 0 \tag{5.11}$$

$$\frac{D\eta}{Dt} = -\frac{\partial E(\rho, S)}{\partial S} = 0 \tag{5.12}$$

where

$$S = S_0(\boldsymbol{\xi}), \qquad \rho = \frac{m\rho_0(\boldsymbol{\xi})}{Q}, \qquad Q = \frac{\partial(\mathbf{x})}{\partial(\boldsymbol{\xi})}. \tag{5.13}$$

The system of five equations (5.10)-(5.12) is a system for five dependent variables $\mathbf{x}, \varphi, \eta$, considered as functions of four independent variables $t, \boldsymbol{\xi}$. Five indefinite functions $\mathbf{g}, \rho_0, S_0$ of $\boldsymbol{\xi}$ are determined from initial and boundary conditions. In particular, if there is no inflow of the fluid and the initial values for $\mathbf{x}, \varphi, \eta$ are given in the form

$$\mathbf{x}(0, \boldsymbol{\xi}) = \boldsymbol{\xi}, \tag{5.14}$$

$$\varphi(0, \boldsymbol{\xi}) = 0, \qquad \eta(0, \boldsymbol{\xi}) = 0, \tag{5.15}$$

it follows from (5.10), (5.13), that

$$S(0,\boldsymbol{\xi}) = S_0(\boldsymbol{\xi}), \qquad \rho(0,\boldsymbol{\xi}) = \rho_0(\boldsymbol{\xi}), \qquad \frac{D\mathbf{x}}{Dt}(0,\boldsymbol{\xi}) = \mathbf{v}_{\text{in}}(\boldsymbol{\xi}) = \mathbf{g}(\boldsymbol{\xi}) \tag{5.16}$$

In the Lagrangian description the initial conditions (5.14) for the position of a particle labeled by $\boldsymbol{\xi}$ look quite reasonable. From physical point of view a necessity of initial conditions for the particle position does not raise doubts. The system of hydrodynamic equations (5.10)-(5.12) in the Lagrangian form looks as partly integrated system. Indeed, the system (1.1)-(1.4) written in the independent Lagrangian coordinates $\boldsymbol{\xi}$ for six dependent variables $\mathbf{x} = \mathbf{x}(t,\boldsymbol{\xi}), \mathbf{v} = \mathbf{v}(t,\boldsymbol{\xi})$ has the form

$$\frac{Dx^\alpha}{Dt} = v^\alpha, \qquad m\frac{Dv^\alpha}{Dt} = -\rho_0(\boldsymbol{\xi})^{-1} X_{\alpha,\beta}\frac{\partial}{\partial \xi_\beta}\frac{\partial}{\partial \rho}[\rho^2 E(\rho, S_0(\boldsymbol{\xi}))], \qquad \alpha = 1,2,3 \tag{5.17}$$

where $X_{\alpha,\beta}$ and $\rho$ are functions of $\boldsymbol{\xi}$, $x^\alpha$ and $x^{\alpha,\beta} \equiv \partial x^\alpha/\partial \xi^\beta$ which are defined by relations (5.1), (5.2), (5.13). The six order system (5.17) contains only two indefinite functions $\rho_0(\boldsymbol{\xi})$ and $S_0(\boldsymbol{\xi})$ describing initial values of density and entropy. Initial values of velocity $\mathbf{v}(0,\boldsymbol{\xi})$ and position $\mathbf{x}(0,\boldsymbol{\xi})$ are given by initial conditions

$$\mathbf{x}(0,\boldsymbol{\xi}) = \boldsymbol{\xi}, \qquad \mathbf{v}(0,\boldsymbol{\xi}) = \mathbf{v}_{\text{in}}(\boldsymbol{\xi}) \tag{5.18}$$

The relation (5.7) is an integral of (5.17). It satisfies the equations (5.17) for any functions $\mathbf{g}$ in virtue of equations (5.10)-(5.12), although this circumstance is not evident directly.

Note that the curtailed system (1.1)-(1.3) cannot be written in the Lagrangian form directly, because it does not contain any reference to Lagrangian coordinates $\boldsymbol{\xi}$. To introduce $\boldsymbol{\xi}$, it is necessary to append equations (1.4). Then fifth order system (1.1)-(1.3) turns to the complete eight order system (1.1)-(1.4). Equations (1.1) and (1.3) can be integrated on the basis of (1.4) in the form (1.9), (1.10). The remaining equations (1.2), (1.4) constitute the sixth order system which can be written in the Lagrangian form (5.17).

# 6 Incompressible fluid

In the special case of the incompressible fluid, it should set $\rho = \rho_0 =$const in the action (3.20) and introduce new variable

$$\mathbf{v} = \mathbf{j}/\rho_0, \qquad \rho_0 = \text{const} \tag{6.1}$$

It is easy to verify that $\eta = \eta(\boldsymbol{\xi})$, $S = S_0(\boldsymbol{\xi})$, and the last term of (3.20) can be incorporated in the term $j^k g^\alpha(\boldsymbol{\xi})\partial_k \xi_\alpha$. Thus, the action for the incompressible fluid looks as follows

$$\mathcal{A}_{\text{E}}[\mathbf{v},\boldsymbol{\xi},\varphi] = \rho_0 \int \{\frac{\mathbf{v}^2}{2} - \mathbf{v}\nabla\varphi - g^\alpha(\boldsymbol{\xi})\partial_0\xi_\alpha - g^\alpha(\boldsymbol{\xi})\mathbf{v}\nabla\xi_\alpha\} dt d\mathbf{x}, \tag{6.2}$$

where $g^\alpha(\boldsymbol{\xi})$ are arbitrary fixed functions of $\boldsymbol{\xi}$.

Variation with respect to $\mathbf{v}$, $\boldsymbol{\xi}$, $\varphi$ gives

$$\delta\mathbf{v}: \qquad \mathbf{v} = \nabla\varphi + g^{\alpha}(\boldsymbol{\xi})\nabla\xi_{\alpha} \tag{6.3}$$

$$\rho_0^{-1}\frac{\delta\mathcal{A}_{\mathrm{E}}}{\delta\xi_{\alpha}} = (g^{\alpha,\beta} - g^{\beta,\alpha})(\partial_0\xi_{\alpha} + \mathbf{v}\nabla\xi_{\alpha}) = 0, \qquad \alpha = 1,2,3 \tag{6.4}$$

$$\rho_0^{-1}\frac{\delta\mathcal{A}_{\mathrm{E}}}{\delta\varphi} = \nabla\mathbf{v} = 0 \tag{6.5}$$

In the general case the condition (3.27) is satisfied. Substituting (6.3) into (6.4) and (6.5), one obtains

$$\partial_0\xi_{\alpha} + [\nabla\varphi + g^{\beta}(\boldsymbol{\xi})\nabla\xi_{\beta}]\nabla\xi_{\alpha} = 0, \qquad \alpha = 1,2,3 \tag{6.6}$$

$$\nabla^2\varphi + g^{\alpha,\beta}(\boldsymbol{\xi})\nabla\xi_{\beta}\nabla\xi_{\alpha} + g^{\alpha}(\boldsymbol{\xi})\nabla^2\xi_{\alpha} = 0 \tag{6.7}$$

The dynamic equation for $\varphi$ does not contain temporal derivative.
Conventional hydrodynamic equations for the incompressible fluid

$$\nabla\mathbf{v} = 0, \qquad \partial_0\mathbf{v} + (\mathbf{v}\nabla)\mathbf{v} = -\frac{\nabla p}{\rho_0} \tag{6.8}$$

are obtained from relations (6.3)-(6.5). Differentiating (6.3) with respect to $t$, one obtains

$$\partial_0\mathbf{v} = \nabla[\partial_0\varphi + g^{\alpha}(\boldsymbol{\xi})\partial_0\xi_{\alpha}] - \Omega^{\alpha\beta}\partial_0\xi_{\beta}\nabla\xi_{\alpha} \tag{6.9}$$

where $\Omega^{\alpha\beta}$ is defined by (1.21). It follows from (6.3), (3.28),(3.29)

$$\mathbf{v}\times(\nabla\times\mathbf{v}) = \Omega^{\alpha\beta}(\boldsymbol{\xi})\nabla\xi_{\beta}(\mathbf{v}\nabla)\xi_{\alpha}. \tag{6.10}$$

In virtue of (6.4) the last term in rhs of (6.9) coincides with rhs of (6.10). Then using the identity (3.32), one obtains

$$\partial_0\mathbf{v} + (\mathbf{v}\nabla)\mathbf{v} = \nabla[\partial_0\varphi + g^{\alpha}(\boldsymbol{\xi})\partial_0\xi_{\alpha} + \frac{1}{2}\mathbf{v}^2] \tag{6.11}$$

The equation (6.11) coincides with the second equation (6.8), provided one uses designation

$$\frac{p}{\rho_0} = -\frac{1}{2}\mathbf{v}^2 - \partial_0\varphi - g^{\alpha}(\boldsymbol{\xi})\partial_0\xi_{\alpha} \tag{6.12}$$

Here the pressure $p$ is determined after solution of the system of hydrodynamic equations (6.3)-(6.5), or (6.8).

If the condition (3.27) is fulfilled, the equation (6.4) may be written in the form (6.6). Then eliminating $\partial_0\xi_{\alpha}$ by means of (6.6) and using the corollary of (6.3)

$$g^{\alpha}(\boldsymbol{\xi})(\mathbf{v}\nabla)\xi_{\alpha} = \mathbf{v}^2 - (\mathbf{v}\nabla)\varphi, \tag{6.13}$$

the relation (6.12) for the pressure can be written in the form

$$\frac{p}{\rho_0} = \frac{1}{2}\mathbf{v}^2 - [\partial_0\varphi + (\mathbf{v}\nabla)\varphi] \tag{6.14}$$

In the case of a irrotational flow, when $\mathbf{v} = \nabla\varphi$, the inequality (3.27) turns to equality and formally the derivation of the expression (6.13) is not founded. Nevertheless the relation (6.14) remains valid in this case, because it turns to the integral

$$\frac{p}{\rho_0} + \frac{1}{2}\mathbf{v}^2 + \partial_0\varphi = 0 \tag{6.15}$$

The relation (6.14) is valid for any flow of incompressible fluid, but it is not an integral. It is a definition. From point of view of the description in terms of hydrodynamic potentials $\varphi$, $\boldsymbol{\xi}$, the relation (6.14) is a definition of the pressure $p$ in terms of dependent variables $\varphi$, $\boldsymbol{\xi}$ and the relation (6.3). From the viewpoint of the curtailed system (6.8) the relation (6.14) is a definition of the function $\varphi$ in terms of variables $\mathbf{v}$, $p$. Nevertheless, this definition is useful for a description of a slightly rotational flow, when the velocity $\mathbf{v}$ can be represented in the form

$$\mathbf{v} = \nabla\varphi + \delta\mathbf{v}, \qquad \mid \delta\mathbf{v} \mid \ll \mid \nabla\varphi \mid, \qquad \delta\mathbf{v} = g^{\alpha}(\boldsymbol{\xi})\nabla\xi_{\alpha}, \tag{6.16}$$

where $\delta\mathbf{v}$ describes a small rotational component of the velocity.

Substituting (6.12) into (6.14), one obtains

$$\frac{p}{\rho_0} = -\partial_0\varphi - \frac{1}{2}(\nabla\varphi)^2 + \frac{1}{2}(\delta\mathbf{v})^2 \tag{6.17}$$

This relation shows that a contribution of the small rotational component $\delta\mathbf{v}$ into the pressure is of the second order $(\delta\mathbf{v})^2$ and this contribution always increases the pressure.

# 7 Slightly Rotational Flow of Incompressible Fluid

It seems to be reasonable to consider a slightly rotational flow as a small correction to an eatablished irrotational flow which can be effectively calculated for a flow around different bodies. Let $u$ be a set of dependent variables, and dependent variables $u_0$ describe an irrotational flow. Let us represent a slightly rotational flow in the form

$$u = u_0 + \varepsilon u_1 + O(\varepsilon^2), \qquad \varepsilon \ll 1 \tag{7.1}$$

where $\varepsilon$ is a small formal parameter which is set to be equal to 1 after calculation. It means that the quantities $u_1$ are considered as small with respect to $u_0$, $u_1/u_0 = O(\varepsilon) \ll 1$. Substituting (7.1) into dynamic equations and neglecting higher order terms, one obtains linear equations for $u_1$. Use of dependent variables $\mathbf{v}$, $p$, satisfying equations (6.8), leads to the following first order approximation equations.

$$\nabla\mathbf{v}_1 = 0, \qquad \partial_0\mathbf{v}_1 + (\nabla\varphi_0\nabla)\mathbf{v}_1 + (\mathbf{v}_1\nabla)\nabla\varphi_0 = -\nabla p_1/\rho_0, \qquad \mathbf{v}_0 = \nabla\varphi_0 \tag{7.2}$$

Dynamic equation for the rotational component $\boldsymbol{\omega}_1 = \nabla\times\mathbf{v}_1$ of the velocity follows from the second equation (7.2)

$$\partial_0\boldsymbol{\omega}_1 + (\nabla\varphi_0\nabla)\boldsymbol{\omega}_1 - (\boldsymbol{\omega}_1\nabla)\nabla\varphi_0 = 0 \tag{7.3}$$

Both equations (7.2) and (7.3) are difficult for a solution. Use of dynamic equations (6.3), (6.6) and (6.7) for hydrodynamic potentials appears to be more effective, because for a fixed velocity $\mathbf{v} = \nabla\varphi_0$ equations (1.4) for $\boldsymbol{\xi}$ are equivalent to the system of ordinary equations (1.6) which can be solved simply enough. Indeed, let the velocity $\mathbf{v} = \nabla\varphi_0$ of the basic established irrotational flow be known, and the orthogonal coordinate system $\xi, \eta, \zeta$ coupled with the flow be determined by the relations

$$\xi = u_\infty^{-1}\varphi_0(\mathbf{x}), \qquad \nabla\xi\cdot\nabla\eta = 0, \qquad \nabla\eta\cdot\nabla\zeta = 0, \qquad \nabla\xi\cdot\nabla\zeta = 0 \tag{7.4}$$

where $u_\infty$ is a constant velocity of the flow at infinity. Then the general solution of the equations (1.4) is written in the form

$$\xi_\alpha = f_\alpha(X - u_\infty t, \eta, \zeta), \qquad \alpha = 1, 2, 3 \tag{7.5}$$

$$X = X(\xi, \eta, \zeta) = \int\limits_0^\xi V_0^{-2}(\xi', \eta, \zeta) d\xi', \qquad V_0^2 = (\nabla\xi)^2 = (\nabla\varphi_0/u_\infty)^2 \tag{7.6}$$

where $f_\alpha$, $\alpha = 1, 2, 3$ are arbitrary functions of three arguments $X - u_\infty t$, $\eta$, $\zeta$. $V_0 = V_0(\xi, \eta, \zeta)$ is a dimensionless velocity of the basic flow considered as a function of coordinates $(\xi, \eta, \zeta)$. This fact can be tested by means of a direct substitution of (7.6) into equation (1.4) with $\mathbf{v} = \nabla\varphi_0$. $X$ can be interpreted as a "distorted" Cartesian coordinate in the direction along the basic flow at infinity. The distortion of the coordinate $X$ is chosen in such a way that the flow were uniform.

Let us use the expansion (7.1) for dependent variables $u = \{\varphi, \boldsymbol{\xi}\}$

$$\mathbf{v} = \mathbf{v}_0 + \varepsilon\mathbf{v}_1 + O(\varepsilon^2) = \nabla\left(\varphi_0 + \varepsilon\varphi_1\right) + \varepsilon g^\alpha(\boldsymbol{\xi})\nabla\xi_\alpha + O(\varepsilon^2) \tag{7.7}$$

$$\mathbf{v}_0 = \nabla\varphi_0, \qquad \mathbf{v}_1 = \nabla\varphi_1 + g^\alpha(\boldsymbol{\xi})\nabla\xi_\alpha \tag{7.8}$$

Substituting (7.7) into (6.7), (6.6) and equating coefficients before equal powers of $\varepsilon$ to zero, one obtains

$$\varepsilon^0: \qquad \nabla^2\varphi_0 = 0 \tag{7.9}$$

$$\varepsilon: \qquad \partial_0\xi_\alpha + \nabla\varphi_0\nabla\xi_\alpha = 0, \qquad \alpha = 1, 2, 3 \tag{7.10}$$

$$\varepsilon: \qquad \nabla^2\xi_\alpha + \nabla[g^\alpha(\boldsymbol{\xi})\nabla\xi_\alpha] = 0, \tag{7.11}$$

Let us note that the labels $\boldsymbol{\xi}$ are calculated only in zeroth approximation. Instead of the expansion of $\boldsymbol{\xi}$ one uses the supposition that the rotational components $g^\alpha(\boldsymbol{\xi})$ of the velocity are small as compared with the irrotational component $\nabla\varphi_0$. General solution of equations (7.10) have the form (7.5). Then (7.11) takes the form

$$\nabla^2\varphi_1 + \nabla\left(A_\alpha\nabla B_\alpha\right) = 0, \qquad g^\alpha(\boldsymbol{\xi})\nabla\xi_\alpha = A_\alpha\nabla B_\alpha \tag{7.12}$$

where $A_\alpha$, $B_\alpha$, $\alpha = 1, 2, 3$ are arbitrary functions of arguments $X - u_\infty t$, $\eta$, $\zeta$. Let $\Sigma$ be the solid boundary of the volume with the fluid. Then equation (7.12) should be solved with the boundary condition

$$\mathbf{n}\nabla\varphi_1|_\Sigma = -\, A_\alpha\mathbf{n}\nabla B_\alpha|_\Sigma \tag{7.13}$$

Note that $\varphi_1$, $\boldsymbol{\xi}$ are hydrodynamic potentials which are not determined inambiguously by the fluid flow. But any set of hydrodynamic potentials determines inambiguously a fluid flow. It is possible one to express the functions $\mathbf{A} = \{A_\alpha\}$, $\mathbf{B} = \{B_\alpha\}$, $\alpha = 1, 2, 3$ via functions $\mathbf{g}(\boldsymbol{\xi}) = \{g^\alpha(\boldsymbol{\xi})\}$, $\alpha = 1, 2, 3$, choosing at $t = 0$ $\boldsymbol{\xi} = \mathbf{x}$. Then the relations

$$\mathbf{B}(X, \eta, \zeta) = \mathbf{x}, \qquad \mathbf{A}(X, \eta, \zeta) = \mathbf{g}(\mathbf{x}) \tag{7.14}$$

determine the form of functions $\mathbf{A}$, $\mathbf{B}$ and the fact that $\mathbf{A}$ is a function of $\mathbf{B}$: $\mathbf{A}$=$\mathbf{g}(\mathbf{B})$. On the other hand the functions $\mathbf{g}(\boldsymbol{\xi})$ are indefinite functions whose form is determined by the initial conditions and by a choice of the labeling $\boldsymbol{\xi}$. The relabeling transformation (1.16) changes the form of the functions $\mathbf{g}(\boldsymbol{\xi})$.

Besides only the rotational component of the vector $A_\alpha \nabla B_\alpha$ is essential, because its irrotational component is compensated by the contribution of $\varphi_1$ into $\mathbf{v}$. Indeed, Let $A_\alpha \nabla B_\alpha$can be represented in the form

$$A_\alpha \nabla B_\alpha = \nabla \Phi + \nabla \times \mathbf{C}, \qquad \nabla \mathbf{C} = 0 \tag{7.15}$$

where $\mathbf{C}$ is some solenoidal vector describing the rotational component of $A_\alpha \nabla B_\alpha$. Substitution of (7.15) into (7.12), (7.13) leads to the relations

$$\nabla^2 \tilde{\varphi}_1 = 0, \qquad \tilde{\varphi}_1 = \varphi_1 + \Phi, \quad \mathbf{n}\nabla\tilde{\varphi}_1|_\Sigma = -\,\mathbf{n}\nabla \times \mathbf{C}|_\Sigma \tag{7.16}$$

In other words, the potential $\varphi_1$ compensates the irrotational component of $A_\alpha \nabla B_\alpha$. As a result the total potential
*tildevarphi*$_1$ satisfies the same equation (7.9) as $\varphi_0$ does. The boundary condition for $\tilde{\varphi}_1$ appears to be qenerated by the rotational component of $\mathbf{v}_1$

$$\mathbf{v} = \nabla(\varphi_0 + \varepsilon\tilde{\varphi}_1) + \nabla \times \mathbf{C} + O(\varepsilon^2), \qquad \nabla^2(\varphi_0 + \varepsilon\tilde{\varphi}_1) = 0 \tag{7.17}$$

$$\mathbf{n}\nabla(\varphi_0 + \varepsilon\tilde{\varphi}_1)|_\Sigma = -\,\varepsilon\mathbf{n} \times \mathbf{C}|_\Sigma \tag{7.18}$$

Solution of the equation (7.12) with the boundary condition (7.13) is written in the form:

$$\varphi_1(\mathbf{x}) = \frac{1}{4\pi}\int\limits_V G(\mathbf{x}, \mathbf{x}')\nabla'[A_\alpha(\mathbf{x}')\nabla' B_\alpha(\mathbf{x}')]d\mathbf{x}' - \frac{1}{4\pi}\oint G(\mathbf{x}, \mathbf{x}')A_\alpha(\mathbf{x}')\nabla' B_\alpha(\mathbf{x}')d\mathbf{S}' \tag{7.19}$$

where $\nabla'$ means the gradient with respect to coordinates $\mathbf{x}'$, and $G(\mathbf{x}, \mathbf{x}')$ is the Green function satisfying the following conditions

$$\nabla'^2 G(\mathbf{x}, \mathbf{x}') = -4\pi\delta(\mathbf{x} - \mathbf{x}'), \qquad \left.\frac{\partial G(\mathbf{x}, \mathbf{x}')}{\partial n'}\right|_{\mathbf{x}' \in \Sigma} = 0 \tag{7.20}$$

where $\partial/\partial n$ means component of the gradient in the direction of the normal, and functions $A_\alpha$, $B_\alpha$ depend only on arguments $X - u_\infty t$, $\eta$, $\zeta$ which are known functions of $\mathbf{x}$. The surface integral is taken over the surface surrounding the volume $V$ with the fluid. This surface includes the surface $\Sigma$ of the body and the infinite surface around $V$. The surface integral in (7.19) contains functions $A_\alpha(\mathbf{x}')$, $B_\alpha(\mathbf{x}')$ determined in the whole volume $V$. It permits to transform the surface integral to the volume one

$$\frac{1}{4\pi}\oint G(\mathbf{x},\mathbf{x}')A_\alpha(\mathbf{x}')\nabla' B_\alpha(\mathbf{x}')d\mathbf{S}' = \frac{1}{4\pi}\int\limits_V \nabla'[G(\mathbf{x},\mathbf{x}')A_\alpha(\mathbf{x}')\nabla' B_\alpha(\mathbf{x}')]d\mathbf{x}' \qquad (7.21)$$

Then the relation (7.19) transforms to

$$\varphi_1(\mathbf{x}) = -\frac{1}{4\pi}\int\limits_V \nabla' G(\mathbf{x},\mathbf{x}')\cdot A_\alpha(\mathbf{x}')\nabla' B_\alpha(\mathbf{x}')d\mathbf{x}' \qquad (7.22)$$

Thus, the slightly rotational flow is described by the relations

$$\mathbf{v} = \nabla\varphi_0 + \varepsilon\left(\nabla\varphi_1 + A_\alpha\nabla B_\alpha\right) + O\left(\varepsilon^2\right) \qquad (7.23)$$

$$\frac{p}{\rho_0} = -\partial_0\left(\varphi_0 + \varepsilon\varphi_1\right) - \frac{1}{2}(\nabla\varphi_0)^2 - \varepsilon\nabla\varphi_0\nabla\varphi_1 + O\left(\varepsilon^2\right) \qquad (7.24)$$

where $A_\alpha$, $B_\alpha$, $\alpha = 1, 2, 3$ are arbitrary functions of arguments $X - u_\infty t$, $\eta$, $\zeta$ and $\varphi_1$ is described by the relation (7.22). Expressions for $p$ is a result of application of (6.17) to the case of the slightly rotational flow.

Relations (7.23), (7.24) satisfy the equations (7.2), (7.3), but a direct derivation of (7.23), (7.24) from (7.2), (7.3) seems to be rather difficult because of terms $(\boldsymbol{\omega}_1\nabla)\nabla\varphi_0$, $(\mathbf{v}_1\nabla)\nabla\varphi_0$. Such terms connect three similar equations of the type (1.4) into united system of three equations and prevent to use equivalence of one equation of the type (1.4) and the system (1.6) of ordinary equations.

The quantities $A_\alpha(\boldsymbol{\xi})$, $B_\alpha(\boldsymbol{\xi})$ and other functions of only $\boldsymbol{\xi}$, for instance $\Omega^{\alpha\beta}(\boldsymbol{\xi})$, defined by (1.21), depend on only arguments $X - u_\infty t$, $\eta$, $\zeta$ and can be regarded as quantities frozen in the basic flow, because they depend on $t$ and $X$ only through $X - u_\infty t$. Gradients of frozen quantities, for instance $A_\alpha\nabla B_\alpha$, are not frozen in the basic flow, in general, because they depend on $t$ and $X$ not only via argument $X - u_\infty t$. They depend on $X$ also through arguments $\nabla\xi\cdot\partial X/\partial\xi = \nabla\xi/\left|\nabla\xi\right|^2$, $\nabla\eta$, $\nabla\zeta$ which depend, in general on $X$, but do not depend on $t$. It is a reason why a description in terms of "frozen quantities" appears to be more effective, than that in terms of the "frozen quantities gradients". Appearance of extraterms in (7.2), (7.3) as compared with (1.4) is connected with the use of the gradients of frozen quantities.

# 8 Wave Function and Kinematic Spin of a Flow

There exists a special complex form of hydrodynamic potentials. In this form the dynamic equation for the irrotational component of the flow is very close to a linear equation.

Idea of the transformation of hydrodynamic potentials is very simple. Let there be an irrotational flow $\mathbf{v} = b\nabla\phi$ of a compressible fluid whose internal energy depends only on $\rho$. Here $b$ is a constant of dimensionality $[b] = [L^2T^{-1}]$ which is introduced to make the potential $\phi$ to be dimensionless. Then dynamic equations (1.1), (1.2) can be integrated and written in the form

$$\partial_0\rho + b\nabla(\rho\nabla\phi) = 0, \qquad b\partial_0\phi + \frac{b^2}{2}(\nabla\phi)^2 = -\frac{\partial}{\partial\rho}[\rho E(\rho)] \tag{8.1}$$

Equation (1.2) is obtained as a gradient of the second equation (8.1). The term $b^2(\nabla\phi)^2/2$ is the principal nonlinear term known as convective nonlinearity term $(\mathbf{v}\nabla)\mathbf{v}$. Introducing the complex variable $\Psi$, this term can be removed. Let

$$\Psi = \sqrt{\rho}e^{i\phi}, \qquad \rho = \bar{\Psi}\Psi, \qquad \mathbf{v} = -\frac{ib}{2\bar{\Psi}\Psi}(\bar{\Psi}\nabla\Psi - \nabla\bar{\Psi}\Psi) \tag{8.2}$$

The bar over symbol means the complex conjugation. Then equations (8.1) are described in the form of one complex equation for the complex dependent variable $\Psi$.

$$ib\partial_0\Psi + \frac{b^2}{2}\nabla^2\Psi = \{\frac{\partial}{\partial\rho}[\rho E(\rho)] + \frac{b^2}{2\sqrt{\rho}}\nabla^2(\sqrt{\rho})\}\Psi, \qquad \rho \equiv \bar{\Psi}\Psi \tag{8.3}$$

Equations (8.1) are imaginary and real components of (8.3). The second term in lhs of (8.3) is linear. It corresponds to the convective term $(\mathbf{v}\nabla)\mathbf{v}$. Now this term contains the second order spatial derivative. It is a price which is paid for the linearity. As a result of the transformation (8.2) the nonlinearity is transmitted from the kinematic term to the dynamic one. The kinematic nonlinearity is stronger, than the dynamic nonlinearity, because the kinematic non-linearity is connected directly with velocities, whereas the dynamic nonlinearity is coupled directly only with forces and with accelerations. Thus, a displacement of the nonlinearity from kinematic terms to dynamic ones weakens the nonlinearity (Rylov, 1989).

The complex variable $\Psi$ is used in quantum mechanics, where it is known as the wave function. The fact that the Schrödinger equation for the wave function describes a irrotational flow of some ideal fluid is well known (Madelung, 1926). For such a fluid the internal energy depends on $\rho$ and $\nabla\rho$. This internal energy has such a form ($E = b^2(\nabla\rho)^2/2\rho^2$, $b = -\hbar/m$) that rhs of (8.3) vanishes, and the equation (8.3) becomes linear.

In the case of incompressible fluid rhs of (8.3) becomes indefinite due to additional constraint. The irrotational flow of the incompressible fluid is described by the equations

$$ib\frac{\partial\Psi}{\partial t} + \frac{b^2}{2}\nabla^2\Psi = \frac{p}{\rho_0}\Psi, \tag{8.4}$$

$$\bar{\Psi}\Psi = \rho_0 = \text{const} \tag{8.5}$$

where the pressure $p$ is considered as some function of time and position. The situation is the same as in the case of equations (6.8), where the value of the pressure $p$ is determined only after the flow is calculated.

The equation (8.4) looks as a linear equation. But in reality it is nonlinear because of the nonlinear constraint (8.5). In the case of a rotational flow some additional nonlinear terms appear. However if the flow is slightly rotational, there is a hope that these additional nonlinear terms, as well the nonlinear constraint (8.5) could be considered as corrections to the irrotational flow described by linear dynamic equation.

Using idea of linearization of the irrotational flow, let us introduce $n$-component complex function $\Psi = \{\Psi_\alpha\}, \alpha = 1, 2, \ldots n$, defining it by the relations

$$\Psi_\alpha = \sqrt{\rho} e^{i\varphi/b} w_\alpha(\boldsymbol{\xi}), \qquad \bar{\Psi}_\alpha = \sqrt{\rho} e^{-i\varphi/b} \bar{w}_\alpha(\boldsymbol{\xi}), \qquad \bar{\Psi}\Psi \equiv \sum_{\alpha=1}^{n} \bar{\Psi}_\alpha \Psi_\alpha \tag{8.6}$$

where the bar over a symbol means the complex conjugate, $w_\alpha(\boldsymbol{\xi})$, $\alpha = 1, 2, \ldots n$ are complex functions of only variables $\boldsymbol{\xi}$, satisfying the relations

$$-\frac{ib}{2} \sum_{\alpha=1}^{n} (\bar{w}_\alpha \frac{\partial w_\alpha}{\partial \xi_\beta} - \frac{\partial \bar{w}_\alpha}{\partial \xi_\beta} w_\alpha) = g^\beta(\boldsymbol{\xi}), \qquad \beta = 1, 2, 3, \qquad \sum_{\alpha=1}^{n} \bar{w}_\alpha w_\alpha = 1 \tag{8.7}$$

$n$ is such a natural number that equations (8.7) admit a solution. Here $b$ is some constant of dimensionality $[b] = [L^2 T^{-1}]$ which is introduced to make the quantities $w_\alpha$ and $\phi = \varphi/b$ dimensionless. The number $n$ and the form of functions $w$ depend on the form of functions $g^\alpha$. But if $w_\alpha$, $\alpha = 1, 2, \ldots n$ is a solution of (8.7), the variables $\Psi_\alpha$ are some functions of $\rho$, $\varphi$, $\boldsymbol{\xi}$, but not of their derivatives, although equations (8.7), determining the transformation (8.6), contain derivatives of functions $w_\alpha$ with respect to $\boldsymbol{\xi}$.

It is easy to verify that $\rho$ and $\rho\mathbf{v}$ with $\mathbf{v}$ defined by (1.11) have the form

$$\rho = \bar{\Psi}\Psi, \qquad \mathbf{j} = \rho\mathbf{v} = -\frac{ib}{2}(\bar{\Psi}\nabla\Psi - \nabla\bar{\Psi}\Psi) - \eta\nabla S\bar{\Psi}\Psi \tag{8.8}$$

Then the variational problem with the action (3.20) appears to be equivalent to the variational problem with the action functional

$$\mathcal{A}[\Psi, \bar{\Psi}, \eta, S] = \int \{\frac{ib}{2}(\bar{\Psi}\partial_0\Psi - \partial_0\bar{\Psi}\cdot\Psi) + \eta\partial_0 S\bar{\Psi}\Psi$$

$$-\frac{1}{2\bar{\Psi}\Psi}[\frac{ib}{2}(\bar{\Psi}\nabla\Psi - \nabla\bar{\Psi}\cdot\Psi) + \eta\nabla S\cdot\bar{\Psi}\Psi]^2 - E[x, \bar{\Psi}\Psi, S]\bar{\Psi}\Psi\} d^4x \tag{8.9}$$

Note that the function $\Psi$ considered as a function of independent variables $\{t, \mathbf{x}\}$ is very indefinite in the sense that the same fluid flow may be described by different $\Psi$-functions. There are two reasons for such an indefiniteness. First, the functions $w_\alpha(\boldsymbol{\xi})$ are not determined uniquely by differential equations (8.7). Second, their arguments $\boldsymbol{\xi}$ as functions of $x$ are determined only to within the transformation (1.16). Description of a fluid in terms of the function $\Psi$ is more indefinite, than the description in terms of the hydrodynamic potentials $\boldsymbol{\xi}$. Information about initial and boundary conditions containing in the functions $\mathbf{g}(\boldsymbol{\xi})$ is lost at the description in terms of the $\Psi$-function. The two-component $\Psi$-function can be obtained directly from the Clebsch variables by means of a proper change of variables (Rylov, 1989). However, one cannot be sure that any flow can be described by two component wave function. According to (8.7) the classification of wave functions over the minimal number of components is connected with the form of functions $g(\boldsymbol{\xi})$ and, hence with the integration of hydrodynamic equations.

Let the function $\Psi$ have $n$ components. Regrouping components of the function $\Psi$ in the action (8.9), one obtains the action in the form

$$\mathcal{A}_E[\Psi,\bar{\Psi},\eta,S]=\int\{\frac{1}{2}[\bar{\Psi}(ib\partial_0+A_0)\Psi+(-ib\partial_0\bar{\Psi}+A_0\bar{\Psi})\Psi]-$$

$$-\frac{1}{2}(ib\nabla\bar{\Psi}-\mathbf{A}\bar{\Psi})(-ib\nabla\Psi-\mathbf{A}\Psi)+$$

$$+\frac{b^2}{4}\sum_{\alpha,\beta=1}^{n}\bar{Q}_{\alpha\beta,\gamma}Q_{\alpha\beta,\gamma}\rho+\frac{b^2}{8\rho}(\nabla\rho)^2-\rho E\}d^4x,\qquad \rho\equiv\bar{\Psi}\Psi \tag{8.10}$$

where

$$A=\{A_0,\mathbf{A}\},\qquad A_0\equiv\eta\partial_0 S,\qquad \mathbf{A}\equiv\eta\nabla S, \tag{8.11}$$

$$Q_{\alpha\beta,\gamma}=\frac{1}{\bar{\Psi}\Psi}\begin{vmatrix}\Psi_\alpha & \Psi_\beta\\ \partial_\gamma\Psi_\alpha & \partial_\gamma\Psi_\beta\end{vmatrix},\qquad \alpha,\beta=1,2,\ldots n\qquad \gamma=1,2,3 \tag{8.12}$$

Corresponding dynamic equations have the form

$$\frac{\delta\mathcal{A}}{\delta\bar{\Psi}_\alpha}=(ib\partial_0+A_0)\Psi_\alpha-\frac{1}{2}(ib\nabla+\mathbf{A})^2\Psi_\alpha-\frac{b^2}{4}\sum_{\mu,\nu=1}^{n}\bar{Q}_{\mu\nu,\gamma}Q_{\mu\nu,\gamma}\Psi_\alpha$$

$$+\frac{b^2}{2}\sum_{\nu=1}^{n}Q_{\alpha\nu,\gamma}\partial_\gamma\bar{\Psi}_\nu+\frac{b^2}{2}\sum_{\nu=1}^{n}\partial_\gamma(Q_{\alpha\nu,\gamma}\bar{\Psi}_\nu)+\frac{\partial}{\partial\rho}[\frac{b^2}{8\rho}(\nabla\rho)^2-\rho E]\Psi_\alpha$$

$$-\nabla(\frac{b^2}{4\rho}\nabla\rho)\Psi_\alpha=0,\qquad \alpha=1,2,\ldots n \tag{8.13}$$

$$\frac{\delta\mathcal{A}}{\delta S}=\partial_i(j^i\eta)-\frac{\partial(\rho E)}{\partial S}=0, \tag{8.14}$$

$$\frac{\delta\mathcal{A}}{\delta\eta}=-\partial_i(j^iS)=0, \tag{8.15}$$

where $j=\{\rho,\mathbf{j}\}=\{j^k\}$, $k=0,1,2,3$ is defined by (8.8).

In the case of the irrotational flow, when $g^\alpha(\boldsymbol{\xi})=\partial\Phi(\boldsymbol{\xi})/\partial\xi_\alpha$ equations (8.7) have a solution for $n=1$, and the function $\Psi$ may have one component. Then all $Q_{\alpha\beta,\gamma}\equiv0$, as it follows from (8.12).

The number $n$ of the $\Psi$-function components in the actions (8.9) and (8.10) is arbitrary. A formal variation of the action with respect to $\Psi_\alpha$ and $\bar{\Psi}_\alpha$, $\alpha=1,2,\ldots n$ leads to $2n$ real dynamic equaitons, but not all of them are independent. There are such combinations of variations $\delta\Psi_\alpha$, $\delta\bar{\Psi}_\alpha$, $\alpha=1,2,\ldots n$ which do not change expressions (8.8) and $\bar{\Psi}\partial_0\Psi-\partial_0\bar{\Psi}\cdot\Psi$. Such combinations of variations $\delta\Psi_\alpha$, $\delta\bar{\Psi}_\alpha$, $\alpha=1,2,\ldots n$ do not change the action (8.9), and corresponding combinations of dynamic equations $\delta\mathcal{A}/\delta\Psi_\alpha=0$, $\delta\mathcal{A}/\delta\bar{\Psi}_\alpha=0$ are identities that associates with a dependence between dynamic equations.

Thus, increasing the number $n$, one increases the number of dynamic equations, but the number of independent dynamic equations remains the same. In such a situation it is important to determine the minimal number $n_m$ of the $\Psi$-function components, sufficient for a solution of equations (8.7) with the given vector field $g^\beta(\boldsymbol{\xi})$ in the space $V_{\boldsymbol{\xi}}$ of the labels $\boldsymbol{\xi}$.

Note that under the relabeling transformations (1.16), the quantity $\mathbf{g}(\boldsymbol{\xi})$ transforms as a vector

$$g^\beta(\boldsymbol{\xi}) \rightarrow \tilde{g}^\beta(\tilde{\boldsymbol{\xi}}) = \frac{\partial \xi_\alpha}{\partial \tilde{\xi}_\beta} g^\alpha(\boldsymbol{\xi}), \qquad \beta = 1, 2, 3 \tag{8.16}$$

It is necessary for the quantities (8.8) and the action (8.9) to be invariant with respect to the transformation (1.16)

Let $\mathcal{G}$ be a set of all vector fields $g^\beta(\boldsymbol{\xi})$ in $V_{\boldsymbol{\xi}}$, and $\mathcal{G}_n$ be a set of such vector fields $g^\beta(\boldsymbol{\xi})$ in $V_{\boldsymbol{\xi}}$ which can be represented in the form

$$g^\beta(\boldsymbol{\xi}) = \sum_{k=1}^{n} \eta_k^2(\boldsymbol{\xi}) \partial \zeta_k(\boldsymbol{\xi}) / \partial \xi_\beta, \qquad \eta_1 \equiv 1 \tag{8.17}$$

where $n$ is a fixed natural number, and the functions $\eta_k$, $\zeta_k$, $k = 1, 2, \ldots n$ are scalars in $V_{\boldsymbol{\xi}}$. Under the relabeling transformation (1.16) the functions (8.17) transform as follows

$$\eta_k(\boldsymbol{\xi}) \rightarrow \tilde{\eta}_k(\tilde{\boldsymbol{\xi}}) = \eta_k(\boldsymbol{\xi}), \qquad \zeta_k(\boldsymbol{\xi}) \rightarrow \tilde{\zeta}_k(\tilde{\boldsymbol{\xi}}) = \zeta_k(\boldsymbol{\xi}), \qquad k = 1, 2, \ldots n$$

$$g^\beta(\boldsymbol{\xi}) \rightarrow \tilde{g}^\beta(\tilde{\boldsymbol{\xi}}) = \frac{\partial \xi_\alpha}{\partial \tilde{\xi}_\beta} g^\alpha(\boldsymbol{\xi}) = \frac{\partial \xi_\alpha}{\partial \tilde{\xi}_\beta} \sum_{k=1}^{n} \eta_k^2(\boldsymbol{\xi}) \frac{\partial \zeta_k(\boldsymbol{\xi})}{\partial \xi_\alpha} = \sum_{k=1}^{n} \tilde{\eta}_k^2(\tilde{\boldsymbol{\xi}}) \frac{\partial \tilde{\zeta}_k(\tilde{\boldsymbol{\xi}})}{\partial \tilde{\xi}_\alpha} \tag{8.18}$$

In other words, a vector field $g^\beta(\boldsymbol{\xi})$ of the form (8.17) transforms into the vector field $\tilde{g}^\beta(\tilde{\boldsymbol{\xi}})$ of the same form (8.17), and the set $\mathcal{G}_n$ is invariant with respect to the group (1.16) of the relabeling transformations.

It is easy to see that

$$\mathcal{G}_{n-1} \subseteq \mathcal{G}_n, \qquad \mathcal{G}_0 = \emptyset, \qquad n = 1, 2, \ldots \tag{8.19}$$

because the $n$th term of the sum (8.17) can be combined with the first one, if $\zeta_n$ is a function of $\eta_n$. Let

$$\mathcal{S}_n = \mathcal{G}_n \backslash \mathcal{G}_{n-1}, \qquad n = 1, 2, \ldots \tag{8.20}$$

Then

$$\mathcal{G} = \bigcup_{s=1}^{s=n_m} \mathcal{S}_s, \qquad \mathcal{S}_l = \emptyset, \qquad l = n_m + 1, n_m + 2, \ldots \tag{8.21}$$

where $n_m$ is the number of nonempty invariant subsets of the set $\mathcal{G}$. Each subset $\mathcal{S}_k$ contains only such vector fields $g^\beta(\boldsymbol{\xi})$ which associate with the $k$-component $\Psi$-function $\Psi = \{\Psi_\alpha\}, \quad \alpha = 1, 2, \ldots k$, having the components

$$\Psi_1 = \left\{ (1 - \sum_{\alpha=2}^{k} \eta_\alpha^2) \rho \right\}^{1/2} \exp[i(\phi + \zeta_1)],$$

$$\Psi_\alpha = \eta_\alpha \sqrt{\rho} \exp[i(\phi + \zeta_\alpha + \zeta_1)], \qquad \alpha = 2, 3, \ldots k \tag{8.22}$$

In particular, the set $\mathcal{S}_1$ associates with an irrotational flow, described by a one-component $\Psi$-function determined by one scalar $\zeta_1$; and the set $\mathcal{S}_2$ associates with a rotational flow described by a two-component $\Psi$-function, determined by three scalar functions $\zeta_1$, $\eta_2$, $\zeta_2$ (Clebsch variables).

Thus, types of the perfect fluid flows can be labeled by invariants of the relabeling group (1.16). This labeling is connected with the minimal number $n_m$ of the wave function components. In the quantum mechanics the minimal number $n_m$ of the wave function components is connected with the spin $s$ of a particle described by this wave function. The relation has the form

$$s = (n_m - 1)/2 \tag{8.23}$$

We shall refer to the quantity $s$ defined by (8.23) as the kinematic spin (k-spin) of the fluid flow. According to this definition the kinematic spin of an irrotational flow is equal to 0. The kinematic spin of the rotational flow $s \geq 1/2$. The k-spin $s = 1/2$ is possible. Is a flow of higher k-spin possible? This interesting question is yet open. It seems to be connected with the complicated problem of knottedness of vortex lines (Moffat, 1969; Bretherton, 1970).

As far as the field $\mathbf{g}(\boldsymbol{\xi})$ is determined by the velocity field $\mathbf{v}_{\text{in}}(\mathbf{x})$, at the initial moment, when $\boldsymbol{\xi} = \mathbf{x}$ and $\mathbf{g}(\mathbf{x}) = \mathbf{v}_{\text{in}}(\mathbf{x})$, the k-spin is determined by the initial velocity $\mathbf{v}_{\text{in}}(\mathbf{x})$. In particular, if the initial velocity can be represented in the Clebsch form (4.31) with $\eta_{\text{in}} = 0$, the k-spin of such a flow is equal to 1/2.

Let us consider a description of the incompressible fluid in terms of the wave function. For the incompressible fluid the action (8.9) has the form

$$\mathcal{A}[\bar{\Psi}, \Psi, p] = \rho_0 \int\int \{\frac{ib}{2}(\bar{\Psi}\partial_0\Psi - \partial_0\bar{\Psi}\cdot\Psi) - \frac{1}{2}\mathbf{v}^2 + P(1 - \bar{\Psi}\Psi)\} dt d\mathbf{x}, \tag{8.24}$$

where $\rho = \rho_0 =$const is the fluid density,

$$\mathbf{v} = -\frac{ib}{2}(\bar{\Psi}\nabla\Psi - \nabla\bar{\Psi}\cdot\Psi) \tag{8.25}$$

is the fluid velocity, and $P$ is the Lagrange multiplier which introduces the constraint

$$\rho/\rho_0 = \bar{\Psi}\Psi = 1. \tag{8.26}$$

The dynamic equations are obtained as a result of a variation with respect to $P$, $\bar{\Psi}$, $\Psi$. Variation of (8.24) with respect to $P$ gives (8.26), and

$$\rho_0^{-1}\frac{\delta\mathcal{A}}{\delta\bar{\Psi}} = ib\partial_0\Psi + i\frac{b}{2}[\mathbf{v}\nabla\Psi + \nabla(\mathbf{v}\Psi)] - P\Psi = 0 \tag{8.27}$$

Let us convolute $\bar{\Psi}$ with (8.27) and take imaginary and real parts of the obtained relation. In virtue of (8.26) one obtains respectively

$$\nabla\mathbf{v} = -\frac{ib}{2}\nabla(\bar{\Psi}\nabla\Psi - \nabla\bar{\Psi}\cdot\Psi) = 0 \tag{8.28}$$

$$P = ib(\bar{\Psi}\partial_0\Psi - \partial_0\bar{\Psi}\cdot\Psi) - \mathbf{v}^2 \tag{8.29}$$

Then the dynamic equation (8.27) takes the form

$$ib(\partial_0\Psi + \mathbf{v}\nabla\Psi) - P\Psi = 0 \tag{8.30}$$

where $\mathbf{v}$ is defined by (8.25). Let us use the expressions (8.6), (8.7) of the wave function via functions $\mathbf{g}(\boldsymbol{\xi})$. Using $\rho = 1$, one derives

$$P = -\partial_0\varphi - g^\alpha(\boldsymbol{\xi})\partial_0\xi_\alpha - \mathbf{v}^2 \tag{8.31}$$

Comparing (8.31) with (6.12), one derives the relation between $P$ and the pressure $p$

$$\frac{p}{\rho_0} = P + \mathbf{v}^2/2 \tag{8.32}$$

# 9 Perturbation Theory for the Flow of $k$-spin 1/2

Let us consider the flow of k-spin 1/2, when the wave function has the form

$$\Psi = \begin{pmatrix} \beta \\ \gamma \end{pmatrix}, \qquad \bar{\Psi} = \begin{pmatrix} \bar{\beta}, \bar{\gamma} \end{pmatrix} \tag{9.1}$$

If the wave function is one-component, k-spin is equal to 0 and the flow is irrotational. If $\gamma \ll \beta$, the flow is slightly rotational. The expanded form of (8.30)

$$ib(\partial_0\beta + \mathbf{v}\nabla\beta) - P\beta = 0, \qquad ib(\partial_0\gamma + \mathbf{v}\nabla\gamma) - P\gamma = 0 \tag{9.2}$$

$$\mathbf{v} = -\frac{ib}{2}(\bar{\beta}\nabla\beta + \bar{\gamma}\nabla\gamma - \nabla\bar{\beta}\cdot\beta - \nabla\bar{\gamma}\cdot\gamma) \tag{9.3}$$

Let the expansion of dynamic variables $\beta$, $\gamma$, $\mathbf{v}$, $P$ have the form

$$\beta = \beta_0 + \epsilon\beta_1 + \epsilon^2\beta_2 + O(\epsilon^3), \qquad \gamma = \epsilon\gamma_1 + \epsilon^2\gamma_2 + O(\epsilon^3),$$

$$P = P_0 + \epsilon P_1 + \epsilon^2 P_2 + O(\epsilon^3), \qquad \mathbf{v} = \mathbf{v}_0 + \epsilon\mathbf{v}_1 + \epsilon^2\mathbf{v}_2 + O(\epsilon^3), \qquad \epsilon \ll 1 \tag{9.4}$$

where $\epsilon$ is a formal small parameter. The expansion is used to within $\epsilon^3$, because it appears that the expansion of variables $\beta$, $\mathbf{v}$, $P$ contains only even powers of $\epsilon$, whereas the expansion of $\gamma$ contains only odd powers of $\epsilon$.

Let us substitute expansions (9.4) into dynamic equations (8.26), (8.28), (8.30) and equate to zero coefficients before different powers of $\epsilon$. In the zeroth order approximation one derives

$$\bar{\beta}_0\beta_0 = 1, \qquad \beta_0 = e^{i\phi_0}, \qquad \mathbf{v}_0 = b\nabla\phi_0 \tag{9.5}$$

where $\phi_0$ is a real variable. Then the equations (8.28) and (8.30) give respectively

$$\nabla^2\phi_0 = 0, \qquad P_0 = -b\partial_0\phi_0 - b^2(\nabla\phi_0)^2, \tag{9.6}$$

Thus in the zeroth approximation we have a problem of an irrotational incompressible flow. This problem is supposed to be solved.

In the first order approximation we have

$$\bar{\beta}_0\beta_1 + \bar{\beta}_1\beta_0 = 0 \qquad \beta_1 = i\phi_1\beta_0 \tag{9.7}$$

where $\phi_1$ is a real quantity. Substituting (9.7) into (8.28), (9.2), one derives

$$\nabla^2\phi_1 = 0, \qquad P_1 = -b\partial_0\phi_1 - b^2\nabla\phi_0\nabla\phi_1, \tag{9.8}$$

As far as the dynamic equation for $\phi_1$ has the same form as for $\phi_0$, one can replace $\phi_0 + \varphi\phi_1$, by one variable $\phi_0$. It means that formally one may set without a loss of generality

$$\phi_1 = 0, \qquad P_1 = 0, \qquad \beta_1 = 0 \tag{9.9}$$

The second equation (9.2) reduces to the form

$$i(\partial_0\gamma_1 + b\nabla\phi_0\nabla\gamma_1) + [\partial_0\phi_0 + b(\nabla\phi_0)^2]\gamma_1 = 0 \tag{9.10}$$

It is easy to see that (9.10) reduces to the equation of the type of (7.10). The general solution of (9.10) has the form

$$\gamma_1 = R_1(X - u_\infty t, \eta, \zeta)\exp[i\phi_0 + i\vartheta_1(X - u_\infty t, \eta, \zeta)] \tag{9.11}$$

where $R_1$, $\vartheta_1$ are arbitrary functions of arguments $X - u_\infty t, \eta, \zeta$ defined by relations (7.4)-(7.6) with $\varphi_0 = b\phi_0$.

Taking into account (9.9), one has in the second order approximation

$$\bar{\beta}_0\beta_2 + \bar{\beta}_2\beta_0 + \bar{\gamma}_1\gamma_1 = 0 \tag{9.12}$$

This equation has the general solution

$$\beta_2 = (-\frac{1}{2}\bar{\gamma}_1\gamma_1 + i\phi_2)\beta_0 \tag{9.13}$$

where $\phi_2$ is a real variable. Equation for $\gamma_2$ has the form

$$i(\partial_0\gamma_2 + b\nabla\phi_0\nabla\gamma_2) + [\partial_0\phi_0 + b(\nabla\phi_0)^2]\gamma_2 = 0 \tag{9.14}$$

which coincides with the equation (9.10) for $\gamma_1$. It means the $\epsilon\gamma_2$ may be included in $\gamma_1$ and without a loss of generality one may set $\gamma_2 = 0$.

The equation (8.25) and (8.28) are written in the form

$$\mathbf{v}_2 = b\nabla\phi_2 + bR_1^2\nabla\vartheta_1, \qquad \nabla^2\phi_2 = -\nabla(R_1^2\nabla\vartheta_1) \tag{9.15}$$

Then the first equation (9.2) gives

$$P_2 = -b(\partial_0\phi_2 + \mathbf{v}_0\nabla\phi_2 + \mathbf{v}_2\nabla\phi_0) \tag{9.16}$$

From (9.5), (9.15) one derives

$$\mathbf{v} = \nabla(\varphi_0 + \epsilon^2\varphi_2) + \epsilon^2 bR_1^2\nabla\vartheta_1 + O(\epsilon^3), \qquad \varphi_0 = b\phi_0, \qquad \varphi_2 = b\phi_2 \tag{9.17}$$

Using (8.32), (9.6), (9.8), (9.9), (9.16), (9.17), one derives after calculations

$$\frac{p}{\rho_0} = -\partial_0(\varphi_0 + \epsilon^2\varphi_2) - \frac{1}{2}(\nabla\varphi_0)^2 - \epsilon^2\nabla\varphi_0\nabla\varphi_2 + O(\epsilon^3), \tag{9.18}$$

Results (9.17), (9.18) agree with (7.23), (7.24), provided one sets $\epsilon^2 = \varepsilon$. The expression (7.23) contains the combination $A_\alpha\nabla B_\alpha$, $\alpha = 1,2,3$, whereas the expression (9.17) contains only two arbitrary functions $R_1$, $\vartheta_1$. The fact is that that (7.23) describes arbitrary slightly rotational flow, whereas the relation (9.17) describes slightly rotational flow of k-spin 1/2. For any flow of k-spin $s \leq 1/2$, the six functions $A_\alpha$, $B_\alpha$, $\alpha = 1,2,3$ reduce to two functions (for instance, $A_1$, $B_1$). But the question about maximal value of k-spin is open now. On one hand, one can prove (see, for instance, appendix to the paper by Eckart, 1960) that any vector field $\mathbf{g}(\mathbf{x})$ in the three-dimensional space can be represented in the Clebsch form

$$\mathbf{g} = \nabla\zeta_1 + \eta_2\nabla\zeta_2 \tag{9.19}$$

On the other hand there are examples of vector fields which cannot be represented in the form (9.19) inside the whole space (Moffat, 1969), if $\zeta_1$, $\eta_2$, $\zeta_2$ are considered as single-valued functions of coordinates.

# 10 Two-dimensional irrotational flow

Let us consider a flow around a circular cylinder of a radius $a$. The cylinder axis is supposed to coincide with the axis of $z$. The flow is directed along the axis of $x$. The basic irrotational flow is described by the potential $\varphi_0$, and by the stream function $\psi$ which are connected with the velocity components $\mathbf{v}_0 = \{v_x, v_y, v_z\}$ by means of relations

$$v_x = \partial_x\varphi_0 = \partial_y\psi, \qquad v_y = \partial_y\varphi_0 = -\partial_x\psi, \qquad v_z = 0 \tag{10.1}$$

We are going to use variables $\varphi, \psi$ as coordinates, and introduce another variables $\xi, \eta, \zeta$ defined by (7.4). In the given case they are determined by relations

$$\xi = u_\infty^{-1}\varphi_0 = bu_\infty^{-1}\phi_0, \qquad \eta = bu_\infty^{-1}\psi, \qquad \zeta = z \tag{10.2}$$

where $u_\infty$ is the velocity of the fluid at infinity. In absence of a circulation one derives for the circular cylinder (see, for instance Lamb, 1932)

$$\xi = (1 + \frac{a^2}{r^2})r\cos\vartheta, \qquad \eta = (1 - \frac{a^2}{r^2})r\sin\vartheta, \tag{10.3}$$

$$r^2 = x^2 + y^2, \qquad \vartheta = \arctan\frac{y}{x}$$

$$\partial_x\xi = \partial_y\eta, \qquad \partial_y\xi = -\partial_x\eta \qquad \nabla^2\xi = 0, \qquad \nabla^2\eta = 0, \tag{10.4}$$

In virtue of (10.4)

$$\nabla\xi\cdot\nabla\eta = 0, \qquad (\nabla\xi)^2 = V_0^2 = (\nabla\eta)^2 = \frac{\partial(\xi,\eta)}{\partial(x,y)} \equiv D_0 \tag{10.5}$$

where $V_o = |\nabla\xi|/u_\infty$ is a dimensionless velocity. For the case of the circular cylinder the determinant

$$V_0^2 = D_0 = 1 + \frac{a^4}{r^4} - \frac{2a^2}{r^2}\cos(2\vartheta) \qquad (10.6)$$

tends to 1, if $r \to \infty$.

Let us represent the variable $X$ in the form

$$X = \xi + \Delta, \qquad \Delta = \Delta(\xi,\eta) = \int\limits_0^\xi [D_0^{-1}(\xi',\eta) - 1]d\xi' \qquad (10.7)$$

Then according to (10.6)

$$D_0^{-1}(\xi,\eta) - 1 = -\frac{a^2/r^2 - 2\cos(2\vartheta)}{r^2/a^2 + a^2/r^2 - 2\cos(2\vartheta)} \qquad (10.8)$$

At $\xi \to \pm\infty$

$$\Delta(\xi,\eta) = \Delta_\pm(\eta) + O(\xi^{-1}), \qquad \xi \to \pm\infty \qquad (10.9)$$

Let us try to find a slightly rotational stationary flow. If the flow is stationary, the functions $A_\alpha$ in (7.23) do not depend on argument $X - u_\infty t$. $B_\alpha$ are linear with respect to $X - u_\infty t$ and do not depend on $\eta, \zeta$. Let the flow do not depend on $\zeta = z$. Then

$$A_1 = f(\eta), \qquad f(\eta) \to 0, \qquad \eta \to \infty \qquad A_2 = A_3 = 0 \qquad (10.10)$$

$$B_1 = X - u_\infty t, \qquad B_2 = B_3 = 0 \qquad (10.11)$$

where $X$ is determined by (7.6). Then according to (7.23), (10.2), one derives

$$\mathbf{v} = u_\infty[1 + \varepsilon f(\eta)]\nabla\xi + \varepsilon f(\eta)\nabla\Delta(\xi,\eta) + \varepsilon\nabla\varphi_1 \qquad (10.12)$$

The potential $\varphi$ is determined by the relation (7.22) which takes the form

$$\varphi_1(\xi,\eta) = -\frac{u_\infty}{4\pi}\int \nabla' G(\mathbf{x},\mathbf{x}') f(\eta')\nabla'\Delta(\xi',\eta')d\mathbf{x}' \qquad (10.13)$$

where $\xi, \eta$ are functions (10.3) of $\mathbf{x} = \{x, y, z\}$, and $\xi', \eta'$ are functions (10.3) of $\mathbf{x}' = \{x', y', z'\}$. Introducing designation

$$\tilde{\nabla} = \{\frac{\partial}{\partial\xi}, \frac{\partial}{\partial\eta}, D_0^{-1/2}\frac{\partial}{\partial\zeta}\} \qquad (10.14)$$

and using (10.4), (10.5), the relation (10.13) reduces to

$$\varphi_1(\xi,\eta) = -\frac{u_\infty}{4\pi}\int \tilde{\nabla}' G(\xi,\eta;\xi',\eta') f(\eta')\tilde{\nabla}'\Delta(\xi',\eta')d\xi' d\eta' \qquad (10.15)$$

where the Green function $G(\xi,\eta;\xi',\eta')$ has the form

$$G(\xi,\eta;\xi',\eta') = -2\ln\sqrt{(\xi-\xi')^2 + (\eta-\eta')^2} - 2\ln\sqrt{(\xi-\xi')^2 + (\eta+\eta')^2} \qquad (10.16)$$

Such a reduction of the Green function is possible only in the case, when $\Delta(\xi, \eta)$ does not depend on $\zeta = z$. Here $\xi$, $\eta$ are functions of $(x, y)$ defined by (10.3), and $\xi'$, $\eta'$ are the same functions of $(x', y')$. Calculation of $D_0$ and of $\partial\Delta/\partial\xi$ in terms of variables $\xi, \eta, \zeta$ gives

$$D_0(\xi, \eta) = \frac{4\sqrt{1+\alpha_1^2-2\alpha_2}}{\alpha_1 + \sqrt{1+\alpha_1^2-2\alpha_2} + \sqrt{2(\alpha_1^2+\alpha_1\sqrt{1+\alpha_1^2-2\alpha_2}-\alpha_2)}} \tag{10.17}$$

$$\frac{\partial\Delta}{\partial\xi} = \frac{\alpha_1 + \sqrt{2(\alpha_1^2+\alpha_1\sqrt{1+\alpha_1^2-2\alpha_2}-\alpha_2)}}{4\sqrt{1+\alpha_1^2-2\alpha_2}} - \frac{3}{4} \tag{10.18}$$

$$\alpha_1 = (\xi^2+\eta^2)/4a^2, \qquad \alpha_2 = (\xi^2-\eta^2)/4a^2 \tag{10.19}$$

Using the relations (7.24), (10.2), (10.5), (10.12), (10.15) and the second relation (7.20) written in the form

$$\frac{\partial G}{\partial \eta}\,(\xi, \eta; \xi', \eta')|_{\eta=0} = 0 \tag{10.20}$$

one obtains for the pressure $p$

$$\frac{p}{\rho_0} = -\frac{1}{2}\,(\nabla\varphi_0)^2 - \varepsilon\nabla\varphi_0\nabla\varphi_1 + O(\varepsilon^2) =$$

$$u_\infty^2 D_0(\xi,\eta)\left[-\frac{1}{2} + \frac{\varepsilon}{4\pi}\frac{\partial}{\partial\xi}\int(\frac{\partial G}{\partial\xi'}\frac{\partial\Delta'}{\partial\xi'} + \frac{\partial G}{\partial\eta'}\frac{\partial\Delta'}{\partial\eta'})f(\eta')d\xi'd\eta'\right] \tag{10.21}$$

$$\Delta' = \Delta(\xi', \eta'), \qquad G = G(\xi, \eta; \xi', \eta') \tag{10.22}$$

Now let us calculate the force $\mathbf{F}$ acting on the cylinder. Components of the external normal $\mathbf{n}$ are described by relations

$$\mathbf{n} = \{n_x, n_y\} = \{\cos\vartheta, \sin\vartheta\} = \{\xi/2a, \sqrt{1-\xi^2/4a^2}\} \tag{10.23}$$

Components of the force $\mathbf{F}$ have the form

$$F_x = -\int\limits_{-\pi}^{\pi} p_0(\vartheta)n_x a d\vartheta = \oint \frac{p(\xi)d\xi}{\sqrt{1-\xi^2/4a^2}}, \qquad F_y = -\int\limits_{-\pi}^{\pi} p_0(\vartheta)n_y a d\vartheta = \oint p(\xi)d\xi \tag{10.24}$$

where the contour integral means the integral in the complex plane of $\xi + i\eta$, taken around the cut $\eta = 0$ between the points $a$ and $-a$. $p(\xi) = p_0(\theta)$ is the pressure on the surface of the cylinder.

Using expression (10.21) for the pressure $p$, one derives

$$F_x = -\rho_0 u_\infty^2 \int\limits_{-\pi}^{\pi} \cos\vartheta a d\vartheta \left\{D_0(\xi, 0)\left[-\frac{1}{2} + \frac{\varepsilon}{4\pi}M(\xi)\right]\right\} + O(\varepsilon^2), \tag{10.25}$$

$$F_y = -\rho_0 u_\infty^2 \int\limits_{-\pi}^{\pi} \sin\vartheta a d\vartheta \left\{ D_0(\xi,0) \left[ -\frac{1}{2} + \frac{\varepsilon}{4\pi} M(\xi) \right] \right\} + O(\varepsilon^2), \qquad (10.26)$$

where

$$M(\xi) = \frac{\partial}{\partial\xi} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \left( \frac{\partial G}{\partial\xi'} \frac{\partial\Delta'}{\partial\xi'} + \frac{\partial G}{\partial\eta'} \frac{\partial\Delta'}{\partial\eta'} \right) f(\eta') d\xi' d\eta', \qquad (10.27)$$

$\xi = 2a\cos\vartheta$ and $D_0(\xi,0) = 4\sin^2\vartheta$. The first term in (10.25), (10.26) vanishes and

$$F_x = -\frac{\varepsilon\rho_0 u_\infty^2}{4\pi} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \left( \frac{\partial I_x}{\partial\xi'} \frac{\partial\Delta'}{\partial\xi'} + \frac{\partial I_x}{\partial\eta'} \frac{\partial\Delta'}{\partial\eta'} \right) f(\eta') d\xi' d\eta' + O(\varepsilon^2), \qquad (10.28)$$

$$F_y = -\frac{\varepsilon\rho_0 u_\infty^2}{4\pi} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \left( \frac{\partial I_y}{\partial\xi'} \frac{\partial\Delta'}{\partial\xi'} + \frac{\partial I_y}{\partial\eta'} \frac{\partial\Delta'}{\partial\eta'} \right) f(\eta') d\xi' d\eta' + O(\varepsilon^2), \qquad (10.29)$$

where

$$I_x(\xi',\eta') = \int\limits_{-\pi}^{\pi} D_0(\xi,0) \frac{\partial G(\xi,0;\xi',\eta')}{\partial\xi} a\cos\vartheta d\vartheta \qquad (10.30)$$

$$I_y(\xi',\eta') = \int\limits_{-\pi}^{\pi} D_0(\xi,0) \frac{\partial G(\xi,0;\xi',\eta')}{\partial\xi} a\sin\vartheta d\vartheta \qquad (10.31)$$

$$G(\xi,0;\xi',\eta') = -2\ln\left[(\xi-\xi')^2 + \eta'^2\right] = -2\ln\left[(2a\cos\vartheta - \xi')^2 + \eta'^2\right] \qquad (10.32)$$

It is easy to verify that $I_y(\xi',\eta') = 0$. Then $F_y = 0$.

$I_x(\xi,\eta)$ is a single-valued function of arguments $\xi^2$ and $\eta^2$ (see details in Appendix). As it follows from (10.6), (10.17), (10.18) $\partial\Delta/\partial\xi$ is a single-valued function of arguments $\xi^2$ and $\eta^2$, and $\Delta$ has the form

$$\Delta(\xi,\eta) = \xi B(\xi^2,\eta^2) \qquad (10.33)$$

where $B$ is a single-valued function of arguments $\xi^2$ and $\eta^2$. Then

$$\frac{\partial I_x}{\partial\xi} \frac{\partial\Delta}{\partial\xi} = \xi C_1(\xi^2,\eta^2), \qquad \frac{\partial I_x}{\partial\eta} \frac{\partial\Delta}{\partial\eta} = \xi C_2(\xi^2,\eta^2) \qquad (10.34)$$

where $C_1$ and $C_2$ are single-valued functions of arguments $\xi^2$ and $\eta^2$. It follows from (10.28) and (10.34) that $F_x$ also vanish in the first order approximation $F_x = O(\varepsilon^2)$. In other words, the D'Alamberian paradox takes place also in the case of the slightly rotational stationary flow.

# 11 Concluding remarks

Basing on the Jacobian technique and on the invariance with respect to the relabeling group, one succeeded to integrate dynamic equtions for the ideal fluid. Three indefinite functions $\mathbf{g}(\boldsymbol{\xi})$ of three arguments arise as a result of this integration.

These functions can be expressed vias initial and boundary conditions, and all essential information on the fluid flow appears to be concentrated in the integrated system of hydrodynamic equations.

Although the integrated system and the curtailed Euler system contain the same number of equations, they differ in the relation that the integrtated system takes into account information contained in the Lin constraints, whereas the Euler system ignores it. This difference is of no importance for irrotational flows, when the fluid properties described by the Kelvin's theorem (and Lin constraints) are inessential, because they are fulfilled automatically. But this difference is essential in the case of irrotational flows.

One should expect that in the case of a compressible fluid the hydrodynamic potentials and the wave function will describe effectively an interaction between the acoustic waves and the slight fluid vorticity.

Integration of dynamic equations generates two ways of descriptions: a description in terms of hydrodynamic potential (DTHP) and a description in terms of wave function (DTWF). Both DTHP and DTWF take into account Lin constraints and appear to be effective in the case of slightly rotational flows. There is a hope they will appear to be effective also in the case of strongly rotational flows. Maybe, an application of DTHP and DTWF will allow to formulate the turbulence problem in a proper way. At any rate DTHP and DTWF seem to be more effective for rotational flows, than the conventional description in terms of velocity, because they take into account information contained in Lin constraints (and Kelvin's theorem). DTHP can take into account even initial and boundary conditions.

The integration of hydrodynamic equations and appearance of the field $\mathbf{g}$ activates the relabeling group. Invariant subsets of this group can be used for a classification of the fluid flows. The field $\mathbf{g}$ appears to be a tool for introducing such attributes of the ideal fluid as the wave function and the spin. These concepts are new for conventional hydrodynamics, although they are well known in quantum mechanics. In some cases it may be useful in the sense that quantum mechanical methods can be used in hydrodynamics and vice versa.

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# 12 Appendix

Let us calculate the expression (10.30)

$$I_x(\xi,\eta) = \int_{-\pi}^{\pi} D_0(2a\cos\vartheta,0)\left.\frac{\partial G(\xi',0;\xi,\eta)}{\partial\xi'}\right|_{\xi'=2a\cos\vartheta} a\cos\vartheta d\vartheta \qquad (A.1)$$

where

$$G(2a\cos\vartheta,0;\xi,\eta) = -2\ln\left[(2a\cos\vartheta-\xi)^2+\eta^2\right] \qquad (A.2)$$

$$\frac{\partial G}{\partial\xi'} = -\frac{1}{2a\sin\vartheta}\frac{\partial G}{\partial\vartheta} = \frac{4(2a\cos\vartheta-\xi)}{(2a\cos\vartheta-\xi)^2+\eta^2} \qquad (A.3)$$

Let us introduce designations

$$\mu=\mu_1+i\mu_2, \qquad \bar{\mu}=\bar{\mu}_1+i\bar{\mu}_2, \qquad \mu_1=\frac{\xi}{2a}, \qquad \mu_2=\frac{\eta}{2a} \qquad (A.4)$$

Then

$$D_0(2a\cos\vartheta,0) = 4\sin^2\vartheta \qquad (A.5)$$

$$I_x(\xi,\eta) = 8\int_{-\pi}^{\pi}\frac{\cos\vartheta-\mu_1}{(\cos\vartheta-\mu_1)^2+\mu_2^2}(1-\cos^2\vartheta)\cos\vartheta d\vartheta =$$

$$4\int_{-\pi}^{\pi}(1-\cos^2\vartheta)\cos\vartheta\left(\frac{1}{\cos\vartheta-\mu_1+i\mu_2}+\frac{1}{\cos\vartheta-\mu_1-i\mu_2}\right)d\vartheta \qquad (A.6)$$

This integral reduces to the form

$$I_x = -16A\pi - 4A\int_{-\pi}^{\pi}\left(\frac{\mu}{\cos\vartheta-\mu}+\frac{\bar{\mu}}{\cos\vartheta-\bar{\mu}}\right)d\vartheta \qquad (A.7)$$

where

$$A = (\mu^2+\bar{\mu}^2+\mu\bar{\mu}-1) = 3\mu_1^2-\mu_2^2-1 \qquad (A.8)$$

Integration of (A.7) leads to

$$I_x = -16A\pi - 4A\left[\frac{\mu}{\sqrt{1-\mu^2}}\ln\left.\frac{\tan\frac{\vartheta}{2}+\frac{1-\mu}{\sqrt{1-\mu^2}}}{\tan\frac{\vartheta}{2}-\frac{1-\mu}{\sqrt{1-\mu^2}}}\right|_{-\pi}^{\pi} + (c.c.)\right] \qquad (A.9)$$

where "(c.c)" means the complex conjugate quantity. Let us introduce the complex quantity

$$\nu = \nu_1+i\nu_2 = \frac{\mu}{\sqrt{1-\mu^2}} = \sqrt{\frac{1-\mu}{1+\mu}} \qquad (A.10)$$

If $\vartheta$ changes from$-\pi$ to $\pi$, the changes of arguments of complex quantities $\tan\frac{\vartheta}{2}+\nu$ and $\tan\frac{\vartheta}{2}-\nu$ are respectively

$$\Delta\arg(\tan\frac{\vartheta}{2}+\nu) = -\pi\mathrm{sgn}(\mathrm{Im}\nu), \qquad \Delta\arg(\tan\frac{\vartheta}{2}-\nu) = \pi\mathrm{sgn}(\mathrm{Im}\nu) \qquad (A.11)$$

Then

$$I_x = -16A\pi + 8A\pi i \left[ \frac{\mu}{\sqrt{1-\mu^2}} - (c.c.) \right] \mathrm{sgn}\nu_2 \qquad (A.12)$$

Let

$$\mu = \mu_1 + i\mu_2 = \cos(\lambda_1 + i\lambda_2) = \cos\lambda \qquad (A.13)$$

where $\mu_1$, $\mu_2$, $\lambda_1$, $\lambda_2$ are real quantities. Then according to (A.10)

$$\nu = \nu_1 + i\nu_2 = \sqrt{\frac{1-\cos\lambda}{1+\cos\lambda}} = \tan\frac{\lambda}{2} = \frac{\sin\lambda_1}{\cos\lambda_1 + \cosh\lambda_2} + i\frac{\sinh\lambda_2}{\cos\lambda_1 + \cosh\lambda_2} \qquad (A.14)$$

It follows from (A.13)

$$\mu_1 = \cos\lambda_1 \cosh\lambda_2, \qquad \mu_2 = \sin\lambda_1 \sinh\lambda_2 \qquad (A.15)$$

Or

$$\sin^2\lambda_1 = |q|^{1/2} - \frac{\mu_1^2 + \mu_2^2 - 1}{2}, \qquad \cos^2\lambda_1 = \frac{\mu_1^2 + \mu_2^2 + 1}{2} - |q|^{1/2} \qquad (A.16)$$

$$\sinh^2\lambda_2 = \frac{\mu_1^2 + \mu_2^2 - 1}{2} + |q|^{1/2}, \qquad \cosh^2\lambda_2 = \frac{\mu_1^2 + \mu_2^2 + 1}{2} + |q|^{1/2} \qquad (A.17)$$

where

$$q \equiv \left( \frac{\mu_1^2 + \mu_2^2 + 1}{2} \right)^2 - \mu_1^2 \equiv \left( \frac{\mu_1^2 + \mu_2^2 - 1}{2} \right)^2 + \mu_2^2 \qquad (A.18)$$

and the square root of the module is supposed to be non-negative.

All functions (A.16), (A.17) are non-negative. They vanish only at the folowing conditions: $\sin^2\lambda_1 = 0$ at $\mu_2 = 0 \wedge \mid \mu_1 \mid \geq 1$, $\cos^2\lambda_1 = 0$ at $\mu_1 = 1$, $\sinh^2\lambda_2 = 0$ at $\mu_2 = 0 \wedge \mid \mu_1 \mid \leq 1$. It means that $\sinh\lambda_2$ is a single-valued function on the complex plane $\mu$ with the cut $\mu_2 = 0 \wedge \mid \mu_1 \mid \leq 1$, and the sign of $\nu_2$

$$\mathrm{sgn}\nu_2 = \mathrm{sgn}\frac{\sinh\lambda_2}{\cos\lambda_1 + \cosh\lambda_2} = \mathrm{sgn}(\sinh\lambda_2) \qquad (A.19)$$

is the same on the whole complex plane $\mu$, because $\mid \cos\lambda_1 \mid \leq \cosh\lambda_2$. $\cos\lambda_1$ may coincide with $-\cosh\lambda_2$ only if $\sin^2\lambda_1 = 0$, and hence $\mu_2 = 0 \wedge \mid \mu_1 \mid \geq 1$. In this case $\cosh^2\lambda_2 = (\mu_1^2 + 1)/2 + \mid (\mu_1^2 - 1)/2 \mid > 1$ for $\mu_1^2 > 1$. $\cosh^2\lambda_2 = 1$ only if $\mu_2 = 0 \wedge \mu_1 = \pm 1$, but these points $\mu = 1$ and $\mu = -1$ lie on the cut and are supposed to be inaccessible. Thus $\nu_2$ has the same sign on the whole complex plane $\mu$ with the cut $\mu_2 = 0 \wedge \mid \mu_1 \mid \leq 1$. Expression in the brackets of (A.12) can be represented as follows

$$\frac{\mu}{\sqrt{1-\mu^2}} - (c.c.) = \frac{\cos\lambda_1}{\sin\lambda_1} - \frac{\cos\bar{\lambda}_1}{\sin\bar{\lambda}_1} = \tan(\frac{\pi}{2} - \lambda) - \tan(\frac{\pi}{2} - \bar{\lambda}) =$$

$$-i\frac{2\sinh\lambda_2\cosh\lambda_2}{\cosh^2\lambda_2 + \sinh^2\lambda_2 - \cos^2\lambda_1 + \sin^2\lambda_1} \qquad (A.20)$$

Using (A.16), (A.17), one obtains

$$\cosh^2\lambda_2+\sinh^2\lambda_2-\cos^2\lambda_1+\sin^2\lambda_1=4\left[\left(\frac{\mu_1^2+\mu_2^2-1}{2}\right)^2+\mu_2^2\right]^{1/2} \qquad (A.21)$$

$$\sinh^2\lambda_2\cosh^2\lambda_2=\left(\frac{\mu_1^2+\mu_2^2-1}{2}\right)^2+\mu_2^2+\left(\frac{\mu_1^2+\mu_2^2}{2}\right)^2$$
$$-\frac{1}{4}+(\mu_1^2+\mu_2^2)\sqrt{\left(\frac{\mu_1^2+\mu_2^2-1}{2}\right)^2+\mu_2^2} \qquad (A.22)$$

(A.18) is positive everywhere on the plane $\mu$ with the cut $\mu_2=0\wedge\mid\mu_1\mid\leq 1$ and

$$I_x=-16A\pi+4A\pi\frac{\sqrt{2\left(\frac{\mu_1^2+\mu_2^2}{2}\right)^2-\mu_1^2+(\mu_1^2+\mu_2^2)\sqrt{\left(\frac{\mu_1^2+\mu_2^2-1}{2}\right)^2+\mu_2^2}}}{\left|\left(\frac{\mu_1^2+\mu_2^2-1}{2}\right)^2+\mu_2^2\right|}\mathrm{sgn}(\sinh\lambda_2) \qquad (A.23)$$

is a single-valued function of arguments $\mu_1^2=\xi^2/4a^2$ and $\mu_2^2=\eta^2/4a^2$.

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